THE SASAKI-RICCI FLOW AND COMPACT SASAKIAN MANIFOLDS OF POSITIVE TRANSVERSE HOLOMORPHIC BISECTIONAL CURVATURE

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1. Introduction

Sasakian geometry is the odd dimensional cousin of the Kähler geometry. Perhaps the most straightforward definition is the following: a Riemannian manifold (M, g) is Sasakian if and only if its metric cone $(C(M) = \mathbb{R}^+ \times M, \bar{g} = dr^2 + r^2 g)$ is Kähler. A Sasakian-Einstein manifold is a Riemannian manifold that is both Sasakian and Einstein. There has been renewed extensive interest recently on Sasakian geometry, especially on Sasakian-Einstein manifolds. Readers are referred to recent monograph Boyer-Galicki [4], and recent survey paper Sparks [60] and the references in for history, background and recent progress of Sasakian geometry and Sasaki-Einstein manifolds.

The *Sasaki-Ricci flow* is introduced by Smoczyk-Wang-Zhang [61] to study the existence of Sasaki-Einstein metrics, more precisely, the *η-Einstein metrics* on Sasakian manifolds. On a Sasakian manifold, there is a one-dimensional foliation structure $\mathcal{F}_\xi$ determined by the *Reeb vector filed* ξ. The transverse structure of the foliation is a *transverse Kähler structure*. And the Sasaki-Ricci flow is just a *transverse Kähler-Ricci flow* which deforms the transverse Kähler structure along its (negative) transverse Ricci curvature on Sasakian manifolds. In particular, short-time and long time existence of the Sasaki-Ricci flow are proved and convergence to an η-Einstein metric is also established when the *basic first Chern class* is negative or null [61], which can be viewed as an odd-dimensional counterpart of Cao's result [12] for the Kähler-Ricci flow.

For Hamilton's Ricci flow, Perelman [47] introduced a functional, called the $\mathcal{W}$ functional, which is monotone along the Ricci flow and has tremendous applications in the Ricci flow. As an application to Kähler geometry, Perelman proved deep results in Kähler-Ricci flow when the first Chern class is positive. For example, he proved that the scalar curvature and the diameter are both uniformly bounded; readers are referred to Sesum-Tian [58] for details. Perelman's results strengthen the belief of Hamilton-Tian conjecture which states that the Kähler-Ricci flow would converge in some fashion when the first Chern class is positive.

In this paper we will study the Sasaki-Ricci flow when the first basic Chern class is positive. First of all, we introduce an analogue of Perelman's $\mathcal{W}$ functional on Sasakian manifolds, which is monotone along the Sasaki-Ricci flow. Then as in the Kähler setting [58], we prove that the *transverse scalar curvature* and the diameter are both uniformly bounded along the Sasaki-Ricci flow. The frame work of the proof is very similar as in the Kähler setting [58]. But there is a major

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difference for $\mathcal{W}$ functional on Sasakian manifolds and Kähler manifolds, namely, the $\mathcal{W}$ functional on a Sasakian manifold is only involved with *basic functions*. While the distance function, for example, is not a basic function. To overcome this difficulty, we introduce a *transverse distance* on Sasakian manifolds, or more precisely, on *quasi-regular* Sasakian manifolds and we will see that the Sasakian structure plays a central role. Nevertheless, this allows us to prove the results along the Sasaki-Ricci flow on quasi-regular Sasakian manifolds, such as the uniform bound of transverse scalar curvature and the diameter along the flow. For *irregular Sasakian structures*, one can approximate the irregular structure by quasi-regular structures, using Rukimbira's results [53]. Using the estimates in [61], the corresponding Sasaki-Ricci flow for irregular structure can be approximated by the Sasaki-Ricci flow for quasi-regular structure. One observation is that the transverse geometric quantities are uniformly bounded under the approximation, such as the transverse scalar curvature. With the help of this approximation, one can show the transverse scalar curvature is uniformly bounded along the Sasaki-Ricci flow for irregular structure. To show the diameter is uniformly bounded, we need the fact that on any compact Sasakian manifold (or more generally, on compact K-contact manifold), there always exists closed orbits of the Reeb vector field ([2], see [54, 55] further development also). Our results give some evidence that the Sasaki-Ricci flow would converge in a suitable sense to a Sasaki-Ricci solition [61]. As a direct consequence, we can prove that the Sasaki-Ricci flow exists for all positive time and converges to a *Sasaki-Ricci soliton* when the dimension is three. This result was announced in [67].

The Frankel conjecture states that a compact Kähler manifold of complex dimension n with positive bisectional curvature is biholomorphic to the complex projective space $\mathbb{CP}^n$. The Frankel conjecture was proved by Siu-Yau [59] using harmonic maps and Mori [44] via algebraic methods. There have been extensive study of Kähler manifolds with positive (or nonnegative) bisectional curvature using the Kähler-Ricci flow, for example to mention [1, 42, 16, 13, 48] to name a few. Since the positivity of bisectional curvature is preserved [1, 42], the Kähler-Ricci flow will converge to a Kähler-Ricci soliton if the initial metric has nonnegative bisectional curvature [13]. Recently Chen-Sun-Tian [15] proved that a compact Kähler-Ricci soliton with positive bisectional curvature is biholomorphic to a complex projective space without using the resolution of the Frankel conjecture. Hence their result together with the previous results gives a proof of the Frankel conjecture via the Kähler-Ricci flow.

It is then very natural to study the Sasakian manifolds with positive curvature in suitable sense. We use the Sasaki-Ricci flow to study Sasakian manifolds when the *transverse holomorphic bisectional curvature* is positive. The transverse (holomorphic) bisectional curvature is defined to be the bisectional curvature of the transverse Kähler structure. Following the proof in the Kähler setting [1, 42], one can show that the positivity of transverse bisectional curvature is preserved along the Sasaki-Ricci flow. It follows that the transverse bisectional curvature is bounded by its transverse scalar curvature, hence it is bounded. It then follows that the Sasaki-Ricci flow will converge to a Sasaki-Ricci soliton in suitable sense. It would then be very interesting to classify Sasaki-Ricci solitons with positive (or nonnegative) transverse bisectional curvature. The success of the classification of

Sasaki-Ricci solitons with positive (or nonnegative) transverse bisectional curvature would lead to classification of Sasakian manifolds with positive (nonnegative) transverse bisectional curvature, parallel to the results in [59, 44, 1, 42].

The organization of the paper is as follows: in Section 2 and Section 3, we summarize definition and some facts of Sasakian manifolds, the transverse Kähler structure and the Sasaki-Ricci flow. In Section 4 we introduce Perelman's $\mathcal{W}$ functional on Sasakian manifolds and prove that it is monotone along the Sasaki-Ricci flow. In Section 5 we prove that the Ricci potential and the transverse scalar curvature are bounded in terms of the diameter along the flow. This section is pretty much the same as in the Kähler setting [58]. In Section 6 we prove that the diameter is uniformly bounded along the Sasaki-Ricci flow if the Sasakian structure is regular or quasi-regular. In Section 7 we use the approximation mentioned above to prove that the diameter is uniformly bounded along the flow if the Sasakian structure is irregular. In Section 8 we study the Sasaki-Ricci flow for the initial metric with positive (nonnegative) transverse bisectional curvature; in particular we prove that the Sasaki-Ricci flow converges to a Sasaki-Ricci soliton with such an initial metric. In appendix we summarize some geometric and topological results of Sasakian manifolds with positive transverse bisectional curvature.

Remark 1.1. *On Mar 29th, 2011, about one day before this article posted on arxiv.org, I found T. Collins posted his paper arXiv:1103.5720 in which he proved that Perelman's results on Kahler-Ricci flow can be generalized to Sasaki-Ricci flow. I saw his paper before I posted this article which has a substantial overlap on generalization of Perelman's results to Sasaki-Ricci flow.*

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2. Sasakian Manifolds

In this section we recall definition and some basic facts of Sasakian manifolds. For instance, the recent monograph [4] is a nice reference for the details. Let (M, g) be a $2n+1$ dimensional smooth Riemannian manifold (M is assumed to be oriented, connected and compact unless specified otherwise), ∇ the Levi-Civita connection of the Riemannian metric g, and let $R(X, Y)$ and Ric denote the Riemannian curvature tensor and the Ricci tensor of g respectively. A Riemannian manifold (M, g) is said to be a Sasakian manifold if and only if the metric cone $\big(C(M) = M \times \mathbb{R}^+, \bar{g} = dr^2 + r^2 g\big)$ is Kähler. M is often identified with the submanifold $M \times \{1\} \subset C(M)$ $(r = 1)$. Let $\bar{J}$ denote the complex structure of $C(M)$ which is compatible with $\bar{g}$. A Sasakian manifold (M, g) inherits a number of geometric structures from the Kähler structure of its metric cone. In particular, the *Reeb vector field* ξ plays a very important role. The vector filed ξ is defined as $\xi = \bar{J}(r\partial_r)$. This gives a 1-form $\eta(\cdot) = r^{-2}\bar{g}(\xi, \cdot)$. We shall use (ξ, η) to denote the corresponding vector filed and 1-form on $M \cong M \times \{1\}$. One can see that

- ξ is a Killing vector field on M and $L_\xi \bar{J} = 0$;
- $\eta(\xi) = 1, \iota_\xi d\eta(\cdot) = d\eta(\xi, \cdot) = 0$;
- the integral curves of ξ are geodesics.

The 1-form η defines a vector sub-bundle $\mathcal{D}$ of the tangent bundle TM such that $\mathcal{D} = ker(\eta)$. There is an orthogonal decomposition of the tangent bundle

$$TM = \mathcal{D} \oplus L\xi,$$

where $L\xi$ is the trivial bundle generalized by ξ. We can then introduce the $(1,1)$ tensor field Φ such that

$$\Phi(\xi) = 0 \text{ and } \Phi(X) = \bar{J}X, X \in \Gamma(\mathcal{D}),$$

where M is identified with $M \times 1 \subset C(M)$. One can see that Φ can also be defined by

$$\Phi(X) = \nabla_X \xi, X \in \Gamma(TM).$$

One can check that Φ satisfies

$$\Phi^2 = -I + \eta \otimes \xi \text{ and } g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

Moreover Φ is compatible with the 2-form $d\eta$

$$d\eta(\Phi X, \Phi Y) = d\eta(X,Y), X, Y \in \Gamma(TM).$$

One can also check that

$$g(X,Y) = \frac{1}{2} d\eta(X, \Phi Y), X, Y \in \Gamma(\mathcal{D}).$$

Hence $d\eta$ defines a symplectic form on $\mathcal{D}$, and η is a contact form on M since $\eta \wedge \left(\frac{1}{2} d\eta\right)^n$ is the volume form of g and is then nowhere vanishing. The following proposition shows some equivalent descriptions of a Sasakian structure. The proof can be found in [4] (Section 7).

Proposition 2.1. *Let (M,g) be a $2n+1$ dimensional Riemannian manifold. Then the following conditions are equivalent and can be used as the definition of a Sasakian structure.*

(1) $\left(C(M) \cong M \times \mathbb{R}^+, \tilde{g} = dr^2 + r^2 g\right)$ *is Kähler.*

(2) *There exists a Killing vector filed ξ of unit length such that the tensor field $\Phi(X) = \nabla_X \xi$ satisfies*
$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X,Y)\xi.$$

(3) *There exists a Killing vector filed ξ of unit length such that the curvature satisfies*
$$R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi.$$

Recall that the Reeb vector field ξ defines a foliation $\mathcal{F}_\xi$ of M through its orbits. There is then a classification of Sasakian structures according to the global property of its orbits. If all the orbits are compact, hence circles, then ξ generates a circle action on M. If the circle action is free, then M is called *regular*. If the circle action is only locally free, then M is called *quasiregular*. On the other hand, if ξ has a non-compact orbit the Sasakian manifold is said to be *irregular*. We conclude this section by introducing the following structure theorem of a quasiregular Sasakian structure (see [4], Theorem 7.1.3).

Theorem 2.2. *Let (M, ξ, η, Φ, g) be a quasi-regular Sasakian manifold with compact leaves. Let $Z = M/\mathcal{F}_\xi$ denote the space of leaves of the Reeb foliation $\mathcal{F}_\xi$. Then*

(1). The leaf space Z is a compact complex orbifold with a Kähler metric h and the Kähler form ω, which defines an integral class $[\omega]$ in $H^2_{orb}(Z,\mathbb{Z})$. The canonical

projection $\pi : (M, g) \to (Z, h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of M diffeomorphic to S^1.

(2). M is the total space of a principle S^1-orbibundle over Z with connection 1-form η whose curvature $d\eta = \pi^*\omega$.

(3). If the foliation $\mathcal{F}_\xi$ is regular, then M is the total space of a principle S^1-bundle over the Kähler manifold (Z, h).

3. Transverse Kähler Structure and the Sasaki-Ricci Flow

In this section we recall the *transverse Kähler structure* of Sasakian manifolds, which is a special case of transverse Riemannian structure. We shall introduce the transverse Kähler structure both globally (coordinate free) and locally (in local coordinates). These two description are equivalent and readers are referred to [4], Section 2.5 for some more details about transverse Riemannian structure.

Let M be a Sasakian manifold with (ξ, η, Φ, g). Recall that $\mathcal{D} = ker(\eta)$ is a $2n$ dimensional subbunble of TM. If we denote

$$J := \Phi|_{\mathcal{D}},$$

then J is a complex structure on $\mathcal{D}$ and it is compatible with $d\eta$. Hence $(\mathcal{D}, J, d\eta)$ defines a Kähler metric on $\mathcal{D}$. We define the transverse Kähler metric g^T as

$$g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi Y), X, Y \in \Gamma(\mathcal{D}). \tag{3.1}$$

The metric g^T is related to the Sasakian metric g by

$$g = g^T + \eta \otimes \eta. \tag{3.2}$$

From the transverse metric g^T, we can define a connection on $\mathcal{D}$ by

$$\begin{aligned} \nabla^T_X Y &= (\nabla_X Y)^p, X, Y \in \Gamma(\mathcal{D}) \\ \nabla^T_\xi Y &= [\xi, Y]^p, Y \in \Gamma(\mathcal{D}), \end{aligned} \tag{3.3}$$

where X^p denotes the projection of X onto $\mathcal{D}$. One can check that this connection is the unique torsion free such that $\nabla^T g^T = 0$. The connection ∇^T is called the *transverse Levi-Civita connection.* Note that by Proposition 2.1 (2), one can get that

$$\nabla^T J = 0.$$

We can further define the *transverse curvature operator* by

$$R^T(X, Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z. \tag{3.4}$$

One can easily check that

$$R^T(X, \xi)Y = 0.$$

Also when $X, Y, Z, W \in \Gamma(\mathcal{D})$, we have the following relation

$$R(X, Y, Z, W) = R^T(X, Y, Z, W) - g(\Phi Y, W)g(\Phi X, Z) + g(\Phi X, W)g(\Phi Y, Z). \tag{3.5}$$

The *transverse Ricci curvature* is then defined by

$$Ric^T(X, Y) = \sum_i g\left(R^T(X, e_i)e_i, Y\right) = \sum_i g^T\left(R^T(X, e_i)e_i, Y\right),$$

where $\{e_i\}$ is an orthonormal basic of $\mathcal{D}$. When $X, Y \in \Gamma(D)$, one can get that

$$Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y) = Ric(X, Y) + 2g^T(X, Y). \tag{3.6}$$

Hence for the *transverse scalar curvature* one can get that

$$R^T = R + 2n. \tag{3.7}$$

Definition 3.1. *A Sasakian manifold is η-Einstein if there are two constants λ and μ such that*

$$Ric = \lambda g + \mu \eta \otimes \eta.$$

By Proposition 2.1, we have $Ric(\xi, \xi) = 2m$, hence $\lambda + \mu = 2n$.

Definition 3.2. *A Sasakian manifold is transverse Kähler-Einstein if there is a constant λ such that*

$$Ric^T = \lambda g^T.$$

By (3.6), a Sasakian manifold is η-Einstein if and only if it is a transverse Kähler-Einstein.

Definition 3.3. *A Sasakian manifold (M, g) is Sasakian-Einstein if $Ric = 2ng$.*

Note that a Sasakian manifold which is Sasakian-Einstein is necessary Ricci positive.

We shall also recall the Sasakian structure and its transverse structure on local coordinates. For the details, see [26] (Section 3) for example. Let (M, ξ, η, Φ, g) be the Sasakian metric and let g^T be the transverse Kähler metric. Let U_α be an open covering of M and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ submersions such that

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

is biholomorphic when $U_\alpha \cap U_\beta$ is not empty. One can choose local coordinate charts $(z_1, \cdots, z_n)$ on V_α and local coordinate charts $(x, z_1, \cdots, z_n)$ on $U_\alpha \subset M$ such that $\xi = \partial_x$, where we use the notations

$$\partial_x = \frac{\partial}{\partial x}, \partial_i = \frac{\partial}{\partial z_i}, \bar{\partial}_j = \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial z_{\bar{j}}}.$$

The map $\pi_\alpha : (x, z_1, \cdots, z_n) \to (z_1, \cdots, z_n)$ is then the natural projection. There is an isomorphism, for any $p \in U_\alpha$,

$$d\pi_\alpha : \mathcal{D}_p \to T_{\pi_\alpha(p)} V_\alpha.$$

Hence the restriction of g on $\mathcal{D}$ gives a well defined Hermitian metric g^T_α on V_α since ξ generates isometries of g. One can actually verify that there is a well defined Kähler metric g^T_α on each V_α and

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

gives an isometry of Kähler manifolds (V_α, g^T_α). The collection of Kähler metrics $\{g^T_\alpha\}$ on $\{V_\alpha\}$ can be used as an alternative definition of the transverse Kähler metric. This definition is essentially just the description of a transverse Riemannian geometry of a Riemannian foliation in terms of *Haefliger cocycles*, see [4] Section 2.5 for more details. We shall also use ∇^T_α, $R^T_\alpha(X, Y)$, Ric^T_α and R^T_α for its Levi-Civita connection, the curvature, the Ricci curvature and the scalar curvature. The two definition of the transverse Kähler structure are actually equivalent. We can see that $(\mathcal{D} \otimes \mathbb{C})^{(1,0)}$ is spanned by the vectors of the form $\{\partial_i + a_i \partial_x\}, 1 \leq i \leq n$, where $a_i = -\eta(\partial_i)$. It is clear that

$$d\eta(\partial_i + a_i\partial_x, \overline{\partial_j + a_j\partial_x}) = d\eta(\partial_i, \partial_{\bar{j}}),$$

Hence the Kähler form of g^T_α on V_α is then the same as $\frac{1}{2}d\eta$ restricted on the slice $\{x = constant\}$ in U_α. Moreover for any $p \in U_\alpha$ and $X, Y \in \mathcal{D}_p$, we have

$$g^T(X,Y) = g^T_\alpha(d\pi_\alpha(X), d\pi_\alpha(Y)).$$

Hence $d\pi_\alpha : \mathcal{D}_p \to T_{\pi_\alpha(p)}V_\alpha$ gives the isometry of g^T on $\mathcal{D}_p$ and g^T_α on $T_{\pi_\alpha(p)}V_\alpha$ for any $p \in U_\alpha$. For example one can easily check the following. For $X, Y, Z \in \Gamma(\mathcal{D})$ and $X_\alpha = d\pi_\alpha(X) \in TV_\alpha$,

$$\begin{aligned} d\pi_\alpha\left(\nabla^T_X Y\right) &= \left(\nabla^T_\alpha\right)_{X_\alpha} Y_\alpha = d\pi_\alpha(\nabla_X Y), \\ d\pi_\alpha\left(R^T(X,Y)Z\right) &= R^T_\alpha(X_\alpha, Y_\alpha)Z_\alpha. \end{aligned}$$

In this paper we shall treat the transverse Kähler metric by these two equivalent descriptions. For example, one can then easily verify curvature identities for R^T since R^T_α is actually the curvature of the Kähler metric g^T_α on V_α. And it is very convenient to do local computations using g^T_α on V_α. While when we deal with some global features of g^T, such as integration by parts, we shall use g^T. For simplicity, we shall suppress the index α without emphasis even when we do local computations on V_α.

Definition 3.4. *A p-form θ on M is called basic if*

$$\iota_\xi\theta = 0, L_\xi\theta = 0.$$

Let Λ^p_B be the sheaf of germs of basic p-forms and $\Omega^p_B = \Gamma(S, \Lambda^p_B)$ the set of sections of Λ^p_B.

The exterior differential preserves basic forms. We set $d_B = d|_{\Omega^p_B}$. Thus the subalgebra $\Omega_B(\mathcal{F}_\xi)$ forms a subcomplex of the de Rham complex, and its cohomology ring $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology ring*. In particular, there is a transverse Hodge theory [23, 33, 66]. The transverse Hodge star operator $*_B$ is defined in terms of the usual Hodge star by

$$*_B\alpha = *(\eta \wedge \alpha).$$

The adjoint $d^*_B : \Omega^p_B \to \Omega^{p-1}$ of d_B is defined by

$$d^*_B = - *_B d_B *_B .$$

The *basic Laplacian* operator $\triangle_B = d_B d^*_B + d^*_B d_B$. The space $\mathcal{H}^p_B(\mathcal{F}_\xi)$ of *basic harmonic p-forms* is then defined to be the kernel of $\triangle_B : \Omega^p_B \to \Omega^p_B$. The transverse Hodge Theorem [23] then says that each basic cohomology class has a unique harmonic representative.

When (M, ξ, η, g) is an Sasakian structure, there is a natural splitting of $\Lambda^p_B \otimes \mathbb{C}$ such that

$$\Lambda^p_B \otimes \mathbb{C} = \oplus\Lambda^{i,j}_B,$$

where $\Lambda^{i,j}_B$ is the bundle of type (i, j) basic forms. We thus have the well defined operators

$$\begin{aligned} \partial_B &: \Omega^{i,j}_B \to \Omega^{i+1,j}_B, \\ \bar\partial_B &: \Omega^{i,j}_B \to \Omega^{i,j+1}_B. \end{aligned}$$

Then we have $d_B = \partial_B + \bar\partial_B$. Set $d^c_B = \frac{1}{2}\sqrt{-1}\left(\bar\partial_B - \partial_B\right)$. It is clear that

$$d_B d^c_B = \sqrt{-1}\partial_B\bar\partial_B, d^2_B = (d^c_B)^2 = 0.$$

We have the adjoint operators

$$\partial_B^* = - *_B \circ \bar{\partial}_B \circ *_B : \Omega_B^{i,j} \to \Omega_B^{i-1,j}$$

and

$$\bar{\partial}_B^* = - *_B \circ \partial_B \circ *_B : \Omega_B^{i,j} \to \Omega_B^{i,j-1}$$

of ∂_B and $\bar{\partial}_B$ respectively. For simplicity we shall use $\partial, \bar{\partial}, \bar{\partial}^*$ and ∂^* if there is not confusion. We can define

$$\triangle_{\partial}^B = \partial\partial^* + \partial^*\partial, \triangle_{\bar{\partial}}^B = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

We define the operator $L : \Omega_B^{i,j} \to \Omega_B^{i+1,j+1}$ by

$$L\alpha = \alpha \wedge \frac{1}{2} d\eta,$$

and its adjoint operator $\Lambda : \Omega_B^{i+1,j+1} \to \Omega_B^{i,j}$ by

$$\Lambda = - *_B \circ L \circ *_B$$

As usual, one has

Proposition 3.5. *On a Sasakian manifold, one has*

$$[\Lambda, \partial] = -\sqrt{-1}\bar{\partial}^*, \tag{3.8a}$$

$$[\Lambda, \bar{\partial}] = \sqrt{-1}\partial^*, \tag{3.8b}$$

$$\partial\bar{\partial}^* + \bar{\partial}^*\partial = \partial^*\bar{\partial} + \bar{\partial}\partial^* = 0, \tag{3.8c}$$

$$\triangle_B = 2\triangle_{\partial}^B = 2\triangle_{\bar{\partial}}^B. \tag{3.8d}$$

We shall also define the transverse Laplacian $\triangle^T$ for basic functions and basic forms as

$$\triangle^T = \frac{1}{2} g_T^{i\bar{j}} \left(\nabla_i^T \nabla_{\bar{j}}^T + \nabla_{\bar{j}}^T \nabla_i^T \right)$$

The operator $\triangle^T$ on basic forms is well-defined. This can be seen easily from the description of the transverse Kähler structure in terms of Haefliger cocycles. For basic functions, we have

$$\triangle^T f = g_T^{i\bar{j}} \nabla_i^T \nabla_{\bar{j}}^T f = g_T^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial z_{\bar{j}}} = -\triangle_{\bar{\partial}} f.$$

Now we define a 2-form ρ^T called the transverse Ricci form as follows.

$$\rho_\alpha^T = -\sqrt{-1}\partial\bar{\partial} \log \det(g_\alpha^T).$$

One can see that the pull back forms $\pi_\alpha^* \rho_\alpha^T$ patch together to give a global basic 2-form on M, which is denoted by ρ^T. As in the Kähler case ρ^T is d_B closed and define a basic cohomology class of type $(1,1)$. The basic cohomology class $[\rho^T]$is independent of the choice of transverse Kähler form in the space of Sasaki space. The basic cohomology class $[\rho^T]/2\pi$ is called the basic first Chern class of M, and is denoted by $c_1^B(M)$. If there exists a transverse Kähler Einstein with $Ric^T = \lambda \rho^T$ for some constant λ, then $c_1^B = \lambda [d\eta]_B$. This implies that c_1^B definite depending on the sign of λ and $c_1(D) = 0$ (see [26, 4] for example). As was pointed out by Boyer, Galicki and Matzeu [6], in the negative and zero basic first class case with $c_1(D) = 0$ the results of El Kacimi-Alaoui [22] together with Yau's estimate [69] imply that the existence of transverse Kähler-Einstein metric. The remaining case is $c_1^B = \lambda [d\eta]_B, \lambda > 0$ and this is the case which is related closely to the

Sasaki-Einstein metrics. In this case, Futaki-Ono-Wang [26] proved that there is a Sasaki-Ricci soliton in c_1^B if M is toric in addition; by varying the Reeb vector field, they can prove the existence of a Sasaki-Einstein metric on M.

It is natural to deform one Sasakian structure to another, for example, to find Sasaki-Einstein metrics. There are many natural ways to deform a Sasakian structure, see [7] for example. In the present paper, we focus on the Sasaki-Ricci flow introduced in [61],

$$\frac{\partial}{\partial t} g^T = g^T - \lambda Ric^T, \tag{3.9}$$

where λ is a positive constant if $c_1^B > 0$. By the so-called D-homothetic deformation, which is essentially just a rescaling of the transverse Kähler structure such that the new metric is still Sasakian,

$$\tilde{\eta} = a\eta, \tilde{\xi} = a^{-1}\xi, \tilde{\Phi} = \Phi, \tilde{g} = ag + a(a-1)\eta \otimes \eta,$$

we can assume that $\lambda = 1$. Roughly speaking, the Sasaki-Ricci flow is to deform a Sasakian metric in such a way that the transverse Kähler metric is deformed along its Ricci curvature. In particular, it induces a Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_\alpha^T = g_\alpha^T - Ric_\alpha^T$$

on each V_α. To study the Sasaki-Ricci flow, we are also interested in the space of Sasakian metrics

$$\mathcal{H} = \{\phi \in C_B^\infty(M), \eta_\phi \wedge (d\eta_\phi)^n \neq 0, \eta_\phi = \eta + d_B^c \phi\}.$$

This space is studied by Guan-Zhang [31, 32] and it can be viewed as an analogue of the space of Kähler metrics in a fixed Kähler class [38, 56, 19]. For any $\phi \in \mathcal{H}$, one can define a new Sasaki metric $(\eta_\phi, \xi, K_\phi, g_\phi)$ with the same Reeb vector field ξ such that

$$\eta_\phi = \eta + d_B^c \phi, K_\phi = K - \xi \otimes d_B^c \phi \circ K.$$

In terms of the potential ϕ, one can write the Sasaki-Ricci flow [61] as

$$\frac{\partial \phi}{\partial t} = \log \frac{\det(g_{i\bar{j}}^T + \phi_{i\bar{j}})}{\det(g_{i\bar{j}}^T)} + \phi - F, \tag{3.10}$$

where F satisfies

$$\rho_g^T - d\eta = d_B d_B^c F.$$

4. Perelman's $\mathcal{W}$ Functional on Sasakian Manifolds

In this section we show that Perelman's $\mathcal{W}$ functional [47] has its counterpart on Sasakian manifolds. First we go over Perelman's $\mathcal{W}$ functional on Kähler manifolds. In particular we repeat some computations of $\mathcal{W}$ functional when the metric g is only allowed to vary as Kähler metrics, which would be used when we consider $\mathcal{W}$ functional on Sasakian manifolds. On Sasakian manifolds, a natural requirement is that the functions appear in $\mathcal{W}$ functional are basic. This requirement fits the Sasakian structure very well and $\mathcal{W}$ functional behaves well under the Sasaki-Ricci flow. More or less, it can be viewed as Perelman's $\mathcal{W}$ functional for the transverse Kähler structure. Certainly there are some different features in the Sasakian setting. For example, the Sasakian structure is not preserved under scaling and the $\mathcal{W}$ functional on Sasakian manifolds is not scaling invariant either.

First we recall the Perelman's $\mathcal{W}$ functional when (M, g) is assumed to be a compact Kähler manifold of complex dimension n. Recall the $\mathcal{W}$ functional, for $\tau > 0$,

$$\mathcal{W}(g, f, \tau) = \int_M \left(\tau(R + |\nabla f|^2) + f\right) e^{-f} \tau^{-n} dV, \tag{4.1}$$

where $f \in C^\infty(M)$ satisfies the constraint

$$\int_M e^{-f} \tau^{-n} dV = 1.$$

Let $z = (z_1, z_2, \cdots, z_n)$ be a holomorphic coordinate chart on M. We use the notation

$$|\nabla f|^2 = g^{i\bar{j}} f_i f_{\bar{j}} = f_i f_{\bar{i}},$$

where we use $g_{i\bar{j}}$ to denote the metric g and $g^{i\bar{j}}$ to denote its inverse.

Suppose $\delta\tau = \sigma$, $\delta f = h$, $\delta g_{i\bar{j}} = v_{i\bar{j}}$ and $v = g^{i\bar{j}} v_{i\bar{j}}$. We also have $\delta dV = v dV$. Note that we assume there is for some function ψ, at least locally, such that $v_{i\bar{j}} = \partial_i \partial_{\bar{j}} \psi$. Hence we will have $v_{i\bar{j}, j} = v_{,i}$.

Proposition 4.1. *We have the following first variation of $\mathcal{W}$,*

$$\begin{aligned} \delta W(g, f, \tau) = &\int_M \left(\sigma(R + \triangle f) - \tau v_{j\bar{i}}(R_{i\bar{j}} + f_{i\bar{j}}) + h\right) \tau^{-n} e^{-f} dV \\ &+ \int_M (2\triangle f - f_i f_{\bar{i}} + R) \left(v - h - n\sigma\tau^{-1}\right) \tau^{-n} e^{-f} dV. \end{aligned} \tag{4.2}$$

Proof. Since we shall use this computation in the Sasakian setting, we include the computations as follows. We compute

$$\begin{aligned} &\delta R = -v_{j\bar{i}} R_{i\bar{j}} - \triangle v, \\ &\delta(g^{i\bar{j}} f_i f_{\bar{j}}) = -v_{j\bar{i}} f_i f_{\bar{j}} + f_i h_{\bar{i}} + h_i f_{\bar{i}}, \\ &\delta(\tau^{-n} e^{-f} dV) = (v - h - n\sigma\tau^{-1}) \tau^{-n} e^{-f} dV, \end{aligned}$$

where we use the following notation for tensors

$$\langle v_{i\bar{j}}, R_{i\bar{j}} \rangle = v_{i\bar{j}} R_{k\bar{l}} g^{i\bar{l}} g^{k\bar{j}} = v_{\bar{i}j} R_{i\bar{j}}.$$

Hence we compute

$$\begin{aligned} \delta\mathcal{W} = &\int_M (\sigma(R + f_i f_{\bar{i}}) + h) \tau^{-n} e^{-f} dV \\ &+ \int_M \tau \left(-v_{j\bar{i}}(R_{i\bar{j}} + f_i f_{\bar{j}}) - \triangle v + f_i h_{\bar{i}} + h_i f_{\bar{i}}\right) \tau^{-n} e^{-f} dV \\ &+ \int_M (\tau(R + f_i f_{\bar{i}}) + f) (v - h - n\sigma\tau^{-1}) \tau^{-n} e^{-f} dV. \end{aligned} \tag{4.3}$$

We can compute, integration by parts,

$$\int_M \triangle v e^{-f} dV = \int_M v(\triangle e^{-f}) dV = \int_M v(-\triangle f + f_i f_{\bar{i}}) e^{-f} dV, \tag{4.4a}$$

$$\int_M (f_i h_{\bar{i}} + h_i f_{\bar{i}}) e^{-f} dV = -2\int_M h(\triangle f - f_i f_{\bar{i}}) e^{-f} dV, \tag{4.4b}$$

$$\int_M v_{j\bar{i}} f_{i\bar{j}} e^{-f} dV = \int_M v_{j\bar{i}} f_i f_{\bar{j}} e^{-f} dV + \int_M v(\triangle f - f_i f_{\bar{i}}) e^{-f} dV. \tag{4.4c}$$

Applying (4.4) to (4.3), then straightforward computations show that

$$\begin{aligned}\delta W(g,f,\tau)=&\int_M \left(\sigma(R+\triangle f)-\tau v_{j\bar i}(R_{i\bar j}+f_{i\bar j})+h\right)\tau^{-n}e^{-f}dV\\&+\int_M (\tau(2\triangle f-f_if_{\bar i}+R)+f)\left(v-h-n\sigma\tau^{-1}\right)\tau^{-n}e^{-f}dV,\end{aligned}$$

where we have uses the fact that

$$\int_M e^{-f}(\triangle f-f_if_{\bar i})dV=0.$$

□

If we choose

$$v_{i\bar j}=\lambda g^T_{i\bar j}-(R_{i\bar j}+f_{i\bar j}), h=v-n\sigma\tau^{-1}=\lambda n-n\sigma\tau^{-1}-(R+\triangle f),$$

where λ is a constant in (4.2). It then follows that

$$\begin{aligned}\delta\mathcal{W}(g,f,\tau)&=\int_M\left(\tau|R_{i\bar j}+f_{i\bar j}|^2-(\lambda\tau-\sigma+1)(R+\triangle f)+\lambda n-n\sigma\tau^{-1}\right)\frac{e^{-f}}{\tau^n}dV\\&=\int_M\left|R_{i\bar j}+f_{i\bar j}-\frac{\lambda\tau-\sigma+1}{2\tau}g_{i\bar j}\right|^2\tau^{-n+1}e^{-f}dV-\frac{n}{4\tau}(\lambda\tau-\sigma-1)^2.\end{aligned}$$

We choose $\sigma=\lambda\tau-1$, then we get

$$\delta\mathcal{W}(g,f,\tau)=\int_M\left|R_{i\bar j}+f_{i\bar j}-\tau^{-1}g_{i\bar j}\right|^2\tau^{-n+1}e^{-f}dV. \tag{4.5}$$

We then consider the following evolution equations, which is the coupled Kähler-Ricci flow,

$$\begin{cases}\dfrac{\partial g_{i\bar j}}{\partial t}&=g_{i\bar j}-R_{i\bar j},\\[2ex]\dfrac{\partial f}{\partial t}&=n\tau^{-1}-R-\triangle f+f_if_{\bar i},\\[2ex]\dfrac{\partial\tau}{\partial t}&=\tau-1.\end{cases} \tag{4.6}$$

Proposition 4.2. *Under the evolution equations* (4.6)*, we have*

$$\frac{d\mathcal{W}}{dt}=\int_M\left|R_{i\bar j}+f_{i\bar j}-\tau^{-1}g_{i\bar j}\right|^2\tau^{-n+1}e^{-f}dV+\int_M|f_{ij}|^2\tau^{-n+1}e^{-f}dV. \tag{4.7}$$

Proof. This is Perelman's formula on Kähler manifolds. We will include the computations here since the similar computations are needed for the Sasakian case. By

(4.2) and (4.6), we compute

$$
\begin{aligned}
\frac{d\mathcal{W}}{dt} =& \int_M \left((\tau-1)(R^T+\triangle f)+\tau(R_{i\bar{j}}-g_{i\bar{j}})(R_{\bar{i}j}+f_{\bar{i}j})\right)\tau^{-n}e^{-f}dV\\
&+\int_M \left(n\tau^{-1}-R-\triangle f+f_if_{\bar{i}}\right)\tau^{-n}e^{-f}dV\\
&+\int_M \left(\tau\left(2\triangle f-f_if_{\bar{i}}+R\right)+f\right)(\triangle^T f-f_if_{\bar{i}})\tau^{-n}e^{-f}dV\\
=& \int_M \left|R_{i\bar{j}}+f_{i\bar{j}}-\tau^{-1}g_{i\bar{j}}\right|^2\tau^{-n+1}e^{-f}dV\\
&-\int_M f_{i\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})\tau^{-n+1}e^{-f}dV+\int_M f_if_{\bar{i}}\tau^{-n}e^{-f}dV\\
&+\int_M \left(\tau\left(2\triangle f-f_if_{\bar{i}}+R\right)+f\right)(\triangle f-f_if_{\bar{i}})\tau^{-n}e^{-f}dV.
\end{aligned}
\tag{4.8}
$$

We then observe that

$$
\begin{aligned}
\int_M f(\triangle f-f_if_{\bar{i}})e^{-f}dV &= -\int_M f\triangle(e^{-f})dV\\
&= -\int_M \triangle f e^{-f}dV\\
&= -\int_M f_if_{\bar{i}}e^{-f}dV.
\end{aligned}
\tag{4.9}
$$

We claim that

$$
\begin{aligned}
\int_M f_{i\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})e^{-f}dV =& \int_M \left(2\triangle f-f_if_{\bar{i}}+R\right)(\triangle f-f_if_{\bar{i}})e^{-f}dV\\
&-\int_M |f_{ij}|^2e^{-f}dV.
\end{aligned}
\tag{4.10}
$$

The proof is complete by (4.8), (4.9) and (4.10). To prove (4.10), we compute, integration by parts,

$$
\begin{aligned}
\int_M f_{i\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})e^{-f}dV =& \int_M \left(-f_i(R_{\bar{i}j}+f_{\bar{i}j})_{,\bar{j}}+f_if_{\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})\right)e^{-f}dV\\
=& \int_M \left(-f_i(R_{,\bar{i}}+(\triangle f)_{\bar{i}})+f_if_{\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})\right)e^{-f}dV\\
=& \int_M \left((\triangle f-f_if_{\bar{i}})(R+\triangle f)+f_if_{\bar{j}}(R_{\bar{i}j}+f_{\bar{i}j})\right)e^{-f}dV.
\end{aligned}
\tag{4.11}
$$

We can then compute that

$$
\int_M f_if_{\bar{j}}f_{\bar{i}j}e^{-f}dV = \int_M \left(-f_{ij}f_{\bar{j}}f_{\bar{i}}-\triangle f|\nabla f|^2+|\nabla f|^4\right)e^{-f}dV.
\tag{4.12}
$$

We also have

$$
R_{\bar{i}j}f_i = f_{ij\bar{i}}-f_{i\bar{i}j}.
\tag{4.13}
$$

Hence we get that

$$
\begin{aligned}
\int_M f_if_{\bar{j}}R_{\bar{i}j}e^{-f}dV =& \int_M f_{\bar{j}}(f_{ij\bar{i}}-f_{i\bar{i}j})e^{-f}dV\\
=& \int_M \left(-|f_{ij}|^2+f_{ij}f_{\bar{i}}f_{\bar{j}}+(\triangle f)^2-\triangle f|\nabla f|^2\right)e^{-f}dV.
\end{aligned}
\tag{4.14}
$$

By (4.12) and (4.14), we get hat

$$\int_M f_i f_{\bar{j}}(R_{\bar{i}j} + f_{\bar{i}j})e^{-f}dV = \int_M \left((\triangle f - |\nabla f|^2)^2 - |f_{ij}|^2\right) e^{-f}dV.$$

This with (4.11) proves the claim. □

From (4.7), it follows directly that $\mathcal{W}$ is strictly increasing along the evolution equations (4.6) unless

$$R_{i\bar{j}} + f_{i\bar{j}} - \tau^{-1}g_{i\bar{j}} = 0, f_{ij} = 0,$$

which implies that ∇f is a (real) holomorphic vector field and g is a Kähler-Ricci (gradient-shrinking) soliton.

Now we consider that (M, ξ, η, Φ, g) is a $2n+1$ dimensional Sasakian manifold. Let

$$\mathcal{W}(g, f, \tau) = \int_M \left(\tau(R^T + |\nabla f|^2) + f\right) e^{-f}dV, \tag{4.15}$$

where $f \in C^\infty_B(M)$ (namely $df(\xi)$=0) and satisfies

$$\int_M \tau^{-n}e^{-f}dV = 1. \tag{4.16}$$

For f basic, $|\nabla f|^2 = |\nabla^T f|^2 = g^{i\bar{j}}_T f_i f_{\bar{j}}$. Recall that R^T is the transverse scalar curvature and satisfies

$$R^T = R + 2n.$$

So $\mathcal{W}$ functional in (4.15) can be viewed as Perelman's $\mathcal{W}$ functional restricted on basic functions and transverse Kahler structure. Note that the normalization condition (4.16) is not exactly the same as in Perelman's $\mathcal{W}$ functional; in particular, $\mathcal{W}$ functional is not scaling invariant and is not invariant under D-homothetic deformation either.

When f is basic, one can see that the integrand in (4.15) is only involved with the transverse Kähler structure. Hence when we compute the first variation of $\mathcal{W}$, all the local computations are just mimic the computations as in the Kähler case. Under the Sasaki-Ricci flow, the Reeb vector field and the transverse holomorphic structure are both invariant, and the metrics are bundle-like; we shall see that when one applies integration by parts, the expressions involved behave essentially the same as in the Kähler case. Hence we would get the similar monotonicity for $\mathcal{W}$ functional along the Sasaki-Ricci flow.

One can choose a local coordinate chart $(x, z_1, \cdots, z_n)$ on a small neighborhood U such that (see [29] for instance)

$$\begin{aligned}
&\xi = \partial_x\\
&\eta = dx - \sqrt{-1}\left(G_j dz_j - G_{\bar{j}} dz_{\bar{j}}\right)\\
&\Phi = \sqrt{-1}\left(X_j \otimes dz_j - X_{\bar{j}} dz_{\bar{j}}\right)\\
&g = \eta \otimes \eta + g^T_{i\bar{j}} dz_i \otimes dz_{\bar{j}},
\end{aligned} \tag{4.17}$$

where $G : U \to \mathbb{R}$ is a (local) real basic function such that $\partial_x G = 0$, and we use the notations $G_i = \partial_i G, G_{\bar{j}} = \partial_{\bar{j}} G$, $X_j = \partial_i + \sqrt{-1}G_i\partial_x, X_{\bar{j}} = \bar{X}_i$, and $g^T_{i\bar{j}} = 2G_{i\bar{j}} = 2\partial_i\partial_{\bar{j}}G$. Then $\mathcal{D} \otimes \mathbb{C}$ is spanned by $\{X_i, X_{\bar{j}}\}, 1 \le i, j \le n$. It is clear that

$$\Phi X_i = \sqrt{-1}X_i, \Phi X_{\bar{j}} = -\sqrt{-1}X_{\bar{j}}.$$

We can also compute that

$$[X_i, X_j] = [X_{\bar{i}}, X_{\bar{j}}] = [\xi, X_i] = [\xi, X_{\bar{j}}] = 0$$

and

$$[X_i, X_{\bar{j}}] = -2\sqrt{-1}G_{i\bar{j}}\partial_x.$$

The transverse Kähler metric is given by $g^T = g^T_{i\bar{j}} dz_i \otimes dz_{\bar{j}}$, where $g^T_{i\bar{j}} = g^T(\partial_i, \partial_j) = g^T(X_i, X_j) = 2G_{i\bar{j}}$. We also use $g^{i\bar{j}}_T$ to denote the inverse of $g^T_{i\bar{j}}$.

We consider that the variation of the Sasakian structure which fixes ξ and the transverse holomorphic structure such that $\delta g^T_{i\bar{j}} = v_{i\bar{j}} = v(X_i, X_j) = v(\partial_i, \partial_j)$, and there is a basic function ψ such that $v_{i\bar{j}} = \psi_{i\bar{j}}$. Let $v = g^{i\bar{j}}_T v_{i\bar{j}}$. Hence we have $v_{i\bar{j}},_j = v_{,i}$ (this holds if there exists a local basic function ψ such that $v_{i\bar{j}} = \psi_{i\bar{j}}$). Suppose $\delta f = h \in C^\infty_B(M), \delta\tau = \sigma$, then

Proposition 4.3. *The first variation of $\mathcal{W}$ on Sasakian manifolds is given by*

$$\begin{aligned}\delta W(g, f, \tau) = &\int_M \Big(\sigma(R^T + \triangle^T f) - \tau v_{j\bar{i}}(R^T_{i\bar{j}} + f_{i\bar{j}}) + h\Big)\tau^{-n} e^{-f} dV \\ &+ \int_M \left(2\triangle^T f - |\nabla^T f|^2 + R^T\right)\left(v - h - n\sigma\tau^{-1}\right)\tau^{-n} e^{-f} dV.\end{aligned} \tag{4.18}$$

Proof. The proof is almost identical to Proposition 4.1. First of all, since the integrand involves only the transverse Kähler structure, the local computations are exactly the same in terms of transverse Kähler structure. For example, the variation of the transverse scalar curvature takes the form of

$$\begin{aligned}\delta R^T &= \delta(g^{i\bar{j}}_T R^T_{i\bar{j}}) = -\delta(g^{i\bar{j}}_T \partial_i \partial_{\bar{j}} \log\det(g_{i\bar{j}})^T) \\ &= -\triangle^T v - \langle v_{i\bar{j}}, R^T_{i\bar{j}}\rangle_{g^T}.\end{aligned}$$

Hence we need to check that integration by parts as in (4.4) also holds with the similar formula in the Sasakian case. We shall give a proof of (4.4c); while (4.4a) and (4.4b) are quite obvious. To verify (4.4c), we write the integrand as the inner product of two basic forms (see (9.2) for the inner product of two basic forms),

$$\int_M v_{i\bar{j}} f_{j\bar{i}} e^{-f} dV = (v_{i\bar{j}} dz_i \wedge dz_{\bar{j}}, e^{-f}\partial\bar{\partial} f).$$

Let $\Phi = v_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$ and $\Psi = \partial\bar{\partial} h$ be two basic forms. We have

$$(\Phi, \Psi) = \sqrt{-1}(\Lambda\Phi, \triangle_{\bar{\partial}} h). \tag{4.19}$$

Actually, using (3.8b), we can get

$$\begin{aligned}(\Lambda\Phi, \triangle_{\bar{\partial}} h) &= (\Lambda\Phi, \bar{\partial}^* \bar{\partial} h) \\ &= (\bar{\partial}\Lambda\Phi, \bar{\partial} h) \\ &= (\Lambda\bar{\partial}\Phi - [\Lambda, \bar{\partial}]\Phi, \bar{\partial} h) \\ &= (-\sqrt{-1}\partial^*\Phi, \bar{\partial} h) \\ &= -\sqrt{-1}(\Phi, \partial\bar{\partial} h) = -\sqrt{-1}(\Phi, \Psi).\end{aligned} \tag{4.20}$$

Note that $\partial\bar{\partial}(e^{-f}) = -\partial\bar{\partial}fe^{-f} + \partial f \wedge \bar{\partial}fe^{-f}$, we have

$$\begin{aligned}
\int_M v_{i\bar{j}} f_{j\bar{i}} e^{-f} &= \int_M (\Phi, -\partial\bar{\partial}(e^{-f}))dV + \int_M (\Phi, \partial f \wedge \bar{\partial} f e^{-f})dV \\
&= \int_M (\sqrt{-1}\Lambda\Phi, \triangle_{\bar{\partial}}(e^{-f}))dV + \int_M v_{i\bar{j}} f_{\bar{i}} f_j e^{-f} dV \\
&= \int_M (\sqrt{-1}\Lambda\Phi, -\triangle^T(e^{-f}))dV + \int_M v_{i\bar{j}} f_{\bar{i}} f_j e^{-f} dV \\
&= \int_M v(\triangle^T f - |\nabla^T f|^2)e^{-f} dV + \int_M v_{i\bar{j}} f_{\bar{i}} f_j e^{-f} dV,
\end{aligned} \tag{4.21}$$

where we use the fact that $\sqrt{-1}\Lambda\Phi = \Lambda(\sqrt{-1}\Phi) = g_T^{i\bar{j}} v_{i\bar{j}} = v$ and $\triangle^T = -\triangle_{\bar{\partial}}$ for basic functions. □

We then consider the $\mathcal{W}$ functional on the coupled Sasaki-Ricci flow

$$\begin{cases}
\dfrac{\partial g_{i\bar{j}}}{\partial t} &= g^T_{i\bar{j}} - R^T_{i\bar{j}}, \\[2mm]
\dfrac{\partial f}{\partial t} &= n\tau^{-1} - R^T - \triangle f + f_i f_{\bar{i}}, \\[2mm]
\dfrac{\partial \tau}{\partial t} &= \tau - 1.
\end{cases} \tag{4.22}$$

The evolution equation for f in (4.22) is actually a *backward heat equation* where the quantities are computed by the metric g at time t. Note that the Sasaki-Ricci flow always has a long time solution. For any time T, if $f(T) \in C^\infty_B(M)$, we shall prove that the backward heat equation for f in (4.22) has a unique smooth solution $f : [0, T] \to C^\infty_B(M)$ (see Proposition 4.8 below). Under the coupled Sasaki-Ricci flow (4.22), we have

Proposition 4.4. *Under the coupled Sasaki-Ricci flow* (4.22)*, then $\mathcal{W}$ functional on Sasakian manifolds satisfies*

$$\frac{d\mathcal{W}}{dt} = \int_M \left| R^T_{i\bar{j}} + f_{i\bar{j}} - \tau^{-1} g^T_{i\bar{j}} \right|^2 \tau^{-n+1} e^{-f} dV + \int_M |f_{ij}|^2 \tau^{-n+1} e^{-f} dV. \tag{4.23}$$

Proof. Note that we assume that the function f is basic. The computations then follow from Proposition 4.3 and the proof as in Proposition 4.2 once we show the corresponding formula of integration by parts as in (4.11), (4.12) and (4.14) hold. All local computations are the same since the integrand involves only transverse Kähler structure. We then finish our proof by showing the following general formula of integration by parts on Sasakian manifolds (see Proposition 4.5 below). □

Actually one can prove some general formula of integration by parts on Sasakian manifolds when the integrand is involved with basic functions and basic forms. The formula in Proposition 4.5 implies that when the integrand involved only the transverse Kähler structure, basic functions and forms, integration by parts takes the formula as in the Kähler setting, with the corresponding terms involved with the Kähler metric replaced by the transverse Kähler metric. Hence Proposition 4.3 and 4.4 can be proved directly using the computations in Kähler setting (see Proposition 4.1 and 4.2) and Proposition 4.5 below.

Let ϕ be a basic (p,q) form ($\phi \in \Omega_B^{p,q}$) such that $\phi = \phi_{A\bar{B}} dz_A \wedge dz_{\bar{B}}, |A| = p, |B| = q$, where A, B are multi-index such that $A = \alpha_1\alpha_2\cdots\alpha_p$, $B = \beta_1\cdots\beta_q$ and $dz_A = dz_{\alpha_1}\wedge\cdots\wedge dz_{\alpha_p}$, $dz_{\bar{B}} = dz_{\bar{\beta}_1}\wedge\cdots\wedge dz_{\bar{\beta}_q}$. Then the tensor

$$g_T^{i\bar{j}}\nabla_i^T\nabla_{\bar{j}}^T\phi = g_T^{i\bar{j}}\nabla_i^T\nabla_{\bar{j}}^T\phi_{A\bar{B}} dz_A\wedge dz_{\bar{B}}$$

is well defined and it is still a basic form. Again we can see this easily from the description of the transverse Kähler structure in terms of Haefliger cocycles. Then we have,

Proposition 4.5. *Let ϕ and ψ be two basic (p,q) forms, then we have the following formula of integration by parts,*

$$\int_M g_T^{i\bar{j}}\nabla_i^T\nabla_{\bar{j}}^T\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{B\bar{D}}dV = -\int_M g_T^{i\bar{j}}\nabla_{\bar{j}}^T\psi_{A\bar{B}}\overline{\nabla_{\bar{i}}^T\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{B\bar{D}}dV. \tag{4.24}$$

Proof. For simplicity, we only prove that ϕ, ψ are $(1,1)$ forms. The general case can be proved similarly. First denote the connection of the transverse Kähler metric as

$$\Gamma_{ij}^k = g_T^{k\bar{l}}\frac{\partial g_{i\bar{l}}^T}{\partial z_j}.$$

Choose a local coordinate chart $(x, z_1, z_2, \cdots, z_n)$ as in (4.17). Then we have

$$\begin{aligned}
g_T^{i\bar{j}}\nabla_i^T\nabla_{\bar{j}}^T\phi_{A\bar{B}} &= \nabla_i^T\left(g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\right)\\
&= \frac{\partial}{\partial z_i}\left(g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\right) + \Gamma_{ik}^i g_T^{k\bar{j}}\phi_{A\bar{B}} - \Gamma_{iA}^p g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{p\bar{B}},\\
\overline{\nabla_{\bar{i}}^T\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D} &= \nabla_i^T\left(\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\right)\\
&= \frac{\partial}{\partial z_i}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D} + \Gamma_{ik}^A\overline{\psi_{C\bar{D}}}g_T^{k\bar{C}}g_T^{\bar{B}D},\\
dV &= c\det(g_{i\bar{j}}^T)dx\wedge dZ\wedge d\bar{Z},
\end{aligned} \tag{4.25}$$

where

$$dZ = dz_1\wedge\cdots\wedge dz_n, d\bar{Z} = dz_{\bar{1}}\wedge\cdots\wedge dz_{\bar{n}}, c = (\sqrt{-1})^n\frac{n!}{2^n}$$

Now define a vector field

$$X = g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\frac{\partial}{\partial z_i}.$$

One can check directly that X is globally defined on M. Hence we can get a $2n$ form α on M such that

$$\alpha = \iota_X dV = c(-1)^i g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\det(g_{i\bar{j}}^T)dx\wedge dz_1\wedge\cdots\hat{dz_i}\cdots dz_n\wedge d\bar{Z}.$$

Then straightforward computation shows that

$$d\alpha = c\frac{\partial}{\partial z_i}\left(g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\det(g_{i\bar{j}}^T)\right)dx\wedge dZ\wedge d\bar{Z}.$$

Hence

$$\int_M\frac{\partial}{\partial z_i}\left(g_T^{i\bar{j}}\nabla_{\bar{j}}^T\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\det(g_{i\bar{j}}^T)\right)dx\wedge dZ\wedge d\bar{Z} = 0.$$

It follows that

$$
\begin{aligned}
&\int_M \frac{\partial}{\partial z_i}\left(g_T^{i\bar{j}}\nabla^T_{\bar{j}}\phi_{A\bar{B}}\right)\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\det(g^T_{i\bar{j}})dx\wedge dZ\wedge d\bar{Z}\\
&= -\int_M g_T^{i\bar{j}}\nabla^T_{\bar{j}}\phi_{A\bar{B}}\frac{\partial}{\partial z_i}\left(\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\det(g^T_{i\bar{j}})\right)dx\wedge dZ\wedge d\bar{Z}\\
&= -\int_M g_T^{i\bar{j}}\nabla^T_{\bar{j}}\phi_{A\bar{B}}\frac{\partial}{\partial z_i}\left(\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\right)\det(g^T_{i\bar{j}})dx\wedge dZ\wedge d\bar{Z}\\
&\quad -\int_M g_T^{i\bar{j}}\nabla^T_{\bar{j}}\phi_{A\bar{B}}\overline{\psi_{C\bar{D}}}g_T^{A\bar{C}}g_T^{\bar{B}D}\frac{\partial\det(g^T_{i\bar{j}})}{\partial z_i}dx\wedge dZ\wedge d\bar{Z}.
\end{aligned}
\tag{4.26}
$$

Note that

$$
\frac{\partial}{\partial z_k}\det(g^T_{i\bar{j}}) = \Gamma^i_{ik}\det(g^T_{i\bar{j}}). \tag{4.27}
$$

Taking (4.25) and (4.27) into account, then (4.24) follows from (4.26). □

Sometimes it is more convenient to write the $\mathcal{W}$ functional as, taking $w = e^{-f/2}$,

$$
\mathcal{W}(g,w,\tau) = \int_M \left(\tau\left(R^T w^2 + 4|\nabla w|^2\right) - w^2\log w^2\right)\tau^{-n}dV, \tag{4.28}
$$

where $w\in C^\infty_B(M)$ such that $dw(\xi)=0$ and

$$
\int_M w^2\tau^{-n}dV = 1.
$$

When w in (4.28) is not assumed to be a basic function, Rothaus' result [52] can be applied to get that there is a nonnegative minimizer w_0. Furthermore, Rothaus [52] showed that a nonnegative minimizer has to be positive everywhere and smooth. Hence one can get a smooth minimizer for (4.1) by letting $f_0 = -2\log w_0$. When w is assumed to be basic, Rothaus' results certainly imply that $\mathcal{W}(g,w,\tau)$ is bounded from below, hence we can define μ functional as

$$
\mu(g,\tau) = \inf_f \mathcal{W}(g,f,\tau), \tag{4.29}
$$

where $df(\xi)=0$ such that $\int_M \tau^{-n}e^{-f}dV = 1$ and it is finite. We could also mimic his proof (for (4.28)) by using a variational approach for $w\in W^{1,2}_B(M)$ instead of $W^{1,2}(M)$, where $W^{1,2}_B(M)$ is the closure of $C^\infty_B(M)$ in $W^{1,2}(M)$. One can actually show that there is a nonnegative minimizer without too much change as in [52], Section 1. We summarize Rothaus' results as follows.

Theorem 4.6 (Rothaus [52])**.** *There is a minimizer $w_0\in W^{1,2}_B(M)$ of (4.28) which is nonnegative. Moreover, w_0 is strictly positive and smooth.*

Proof. First of all, we show that there exists a nonnegative minimizer $w_0\in W^{1,2}_B$. The statement is a special case considered by Rothaus applied to $W^{1,2}_B$ instead of $W^{1,2}$. The proof follows the line in [52] and it is almost identical ([52], Section 1). The only difference is to prove that a minimizer w_0 is actually a weak solution of the equation,

$$
4\triangle w_0 + w_0\log w_0^2 - R^T w_0 + \mu(g,\tau)w_0 = 0. \tag{4.30}
$$

The proof is slightly more involved and we need that R^T is a basic function. we shall prove this in Appendix. Once we show that w_0 is a weak solution of (4.30) and it is bounded, we can actually apply Rothaus' result to show get that w_0

is strictly positive (see Lemma on page 114, [52]); hence we would get a smooth minimizer. □

Actually what we really need is the fact that the μ functional is nondecreasing along the Sasaki-Ricci flow, which would be enough for us in essence and does not really depend on the above theorem. The proof is similar as in Ricci flow case if we replace a minimizer by a minimizing sequence.

Proposition 4.7. *Let τ_0 be a positive constant and*

$$\tau(t) = (\tau_0 - 1)e^t + 1 > 0, t \in [0, T]$$

Then the functional $\mu(g(t), \tau(t))$ is monotone increasing along the Sasaki-Ricci flow on $[0, T]$.

Proof. It is clear that $\tau(t)$ satisfies $\partial_t \tau = \tau - 1$ with $\tau(0) = \tau_0$. Suppose at time T, there is a minimizing sequence $\{f_k(T)\}$ such that

$$\mathcal{W}_k = \mathcal{W}(g(T), f_k(T), \tau(T)) \to \mu(g(T), \tau(T)), k \to \infty.$$

By Proposition 4.8 below, there is a basic function $f_k(t)$ satisfies (4.28). Hence by Proposition 4.4, we have that

$$\mathcal{W}_k(t) = \mathcal{W}(g(t), f_k(t), \tau) \leq \mathcal{W}_k.$$

Hence $\mu(g(t), \tau(t)) \leq \mathcal{W}_k(t) \leq \mathcal{W}_k$. Let $k \to \infty$, we get $\mu(g(t), \tau(t)) \leq \mu(g(T), \tau(T))$. □

Now we consider the backward heat equation

$$\frac{\partial f}{\partial t} = n\tau^{-1} - R^T - \triangle f + |\nabla f|^2, \tag{4.31}$$

where τ is a positive number depending only on time such that $\partial_t \tau = \tau - 1$. The corresponding backward heat equation for w is

$$\frac{\partial w}{\partial t} = -\triangle w + (R^T - n\tau^{-1})w. \tag{4.32}$$

Proposition 4.8. *Suppose the Sasaki-Ricci flow exists in $[0, T]$, then for any $w(T) \in C^\infty(M)$, there is a unique solution $w(t) \in C^\infty(M)$ of (4.32) for $t \in [0, T]$. If $w(T) \geq 0$, then $w(t) \geq 0$; and if in addition $w(T)$ is not identically zero, then $w(t) > 0$ for $t \in [0, T)$. Moreover if $w(T) \in C^\infty_B(M)$, then $w(t) \in C^\infty_B(M)$. As a consequence, then for any $f(T) \in C^\infty_B(M)$, there exists a unique solution $f(t) \in C^\infty_B(M)$ for any $t \in [0, T]$. If $w(T) \in W^{1,2}_B$ is nonnegative, then $w(t) \in C^\infty_B$ and it is nonnegative.*

Proof. The proof is similar as in the Ricci flow case (c.f [34] for example) and f is basic due to the maximum principle. Let $s = T - t$ and $\tilde{f}(s) = \exp(-f(s))$, then one can compute that

$$\frac{\partial \tilde{f}}{\partial s} = \triangle_{g(s)} \tilde{f} - (R^T - n\tau^{-1})\tilde{f},$$

such that $\tilde{f}(0) \in C^\infty_B(M)$. This is a linear equation for $\tilde{f}$ and there exists a unique smooth solution if $\tilde{f}(0)$ is smooth. Now if $\tilde{f}(0) \geq 0$, then $\tilde{f} \geq 0$ and if it is positive

at one point in addition, we claim that $\tilde{f} > 0$ for any $s \in [0, T]$. Suppose $s_0 \in (0, T]$ such that $s_0 = T - t_0$. Let $h(t)$ satisfies the heat equation

$$\frac{\partial h}{\partial t} = \triangle h$$

on $(t_0, T]$ such that $\lim_{t \to t_0} h \to \delta_p$ for any $p \in M$, where δ_p is the delta function. Consider $\tilde{f}(T-t)$ for $t \in [t_0, T]$, then by direct computations we can get that

$$\frac{d}{dt}\int_M \tau^{-n}\tilde{f}(T-t)h(t)dV = 0. \tag{4.33}$$

It is clear that $h > 0$ in $(t_0, T]$. Then

$$\tilde{f}(p, T-t_0) = \int_M \tilde{f}(q, T-t)\delta_p(q)dV(q) = \tau^n(t_0)\lim_{t\to t_0}\int_M \tau^{-n}\tilde{f}(T-t)h(t)dV.$$

Hence by (4.33) we get

$$\tilde{f}(p, T-t_0) = \tau^n(t_0)\tau^{-n}(T)\int_M \tilde{f}(0)h(T)dV \geq 0.$$

And it is clear that if $\tilde{f}(0) > 0$ somewhere, then $\tilde{f}(p, T-t_0) > 0$ for any p. Suppose $\tilde{f}(0)$ is basic, then for $\xi\tilde{f}$ we have

$$\frac{\partial(\xi\tilde{f})}{\partial s} = \triangle_{g(s)}(\xi\tilde{f}) - (R^T - n\tau^{-1})(\xi\tilde{f}).$$

since R^T is basic. Hence by the uniqueness of the solution we know that $\xi\tilde{f} \equiv 0$ if $\xi\tilde{f}(0) = 0$. Let $f(T-t) = -\log\tilde{f}(T-t)$, then it satisfies (4.31). If $\tilde{f}(0) \in W^{1,2}$, then by the standard regularity theory for the linear parabolic theory, we have $\tilde{f}(t) \in C^\infty$. □

5. The Ricci Potential and the Scalar Curvature along the Flow

We consider the (transverse) scalar curvature and the (transverse) Ricci potential along the Sasaki-Ricci flow. The following discussions and computation follow closely the Kähler setting [58].

First we show that there is a uniform lower bound on the transverse Ricci potential $u(x, t)$, which is defined by

$$g^T_{i\bar{j}} - R^T_{i\bar{j}} = \partial_i\bar{\partial}_j u \tag{5.1}$$

with the normalized condition

$$\int_M e^{-u}dV_g = 1. \tag{5.2}$$

Note that $u \in C^\infty_B(M)$. We compute

$$\begin{aligned}\partial_t\partial_i\bar{\partial}_j u(x,t) &= g^T_{i\bar{j}} - R^T_{i\bar{j}} + \partial_t\partial_i\bar{\partial}_j\log(g^T_{i\bar{j}}(x,0) + \partial_i\bar{\partial}_j\phi(x,t)) \\ &= \partial_i\bar{\partial}_j(u + \triangle^T u).\end{aligned}$$

Hence there exists a time dependent constant $A(t)$ such that

$$\frac{\partial u}{\partial t} = \triangle^T u + u + A, \tag{5.3}$$

where

$$A = -\int_M ue^{-u}dV_g$$

in view of the normalized condition (5.2).

Lemma 5.1. *$A(t)$ is uniformly bounded.*

Proof. It is clear that $f(t) = te^{-t} \le e^{-1}$ is bounded from above for any $t \in \mathbb{R}$. It then follows that
$$A(t) = -\int_M ue^{-u} dV_g \ge -e^{-1} \int_M dV_g.$$
Note that the volume of (M, ξ, g) is a constant along the flow.

Let $\tau_0 = 1$ (hence $\tau \equiv 1$) and consider the functional $\mu(g(t), 1)$. Hence
$$\mu(g(0), 1) \le \mu(g(t), 1).$$
Note that $R^T = n - \triangle^T u$; hence we get that
$$\mathcal{W}(g(t), u(t), 1) = \int_M (R^T + u_i u_{\bar{i}} + u)e^{-u} dV_g = n + \int_M ue^{-u} dV_g.$$
It follows that
$$\int_M ue^{-u} dV_g \ge \mathcal{W}(g(t), u(t), 1) - n \ge \mu(g(0), 1) - n.$$
It follows that
$$A(t) = -\int_M ue^{-u} dV_g \le -\mu(g(0), 1) + n.$$
□

Proposition 5.2. *The transverse scalar curvature R^T satisfies*
$$\frac{\partial R^T}{\partial t} = \triangle^T R^T - R^T + |Ric^T|^2. \tag{5.4}$$
In particular, R^T is bounded from below.

Proof. A straightforward computation gives (5.4). Note that $R^T \in C^\infty_B(M)$; then $\triangle^T R^T = \triangle R^T$. Hence by the maximum principle R^T is bounded from below. □

Lemma 5.3. *$u(x, t)$ is uniformly bounded from below.*

Proof. By Lemma 5.1 and Proposition 5.2, $-R^T + A(t)$ is bounded from above. Suppose $u(x_0, t_0)$ is very negative at (x_0, t_0) such that
$$u(x_0, t_0) + \max_{x,t}(n - R^T + A) \ll 0.$$
Then we can get that
$$\frac{\partial u}{\partial t} = n - R^T + u + A < 0 \tag{5.5}$$
at (x_0, t_0). It follows that $u(x, t) \le u(x_0, t_0)$ for $t \ge t_0$. Since $u(x, t)$ is smooth, there exists a neighborhood U of x_0 such that for any $x \in U$,
$$u(x, t_0) + \max_{x,t}(n - R^T + A) \ll 0.$$
It then follows from (5.5) that $u(x, t) \le u(x, t_0)$ for any $(x, t) \in U \times [t_0, \infty)$. We can get that
$$u(x, t) \le e^{t-t_0}(C + u(x, t_0)) \text{ by } \frac{\partial u}{\partial t} \le C + u.$$
Hence for $(x, t) \in U \times [t_0, \infty)$,
$$u(x, t) \le -C_1 e^t,$$

where C_1 depends on t_0. Then $\partial\phi/\partial t = u$ implies that

$$\phi(x,t) \le \phi(x,t_0) - C_1 e^{t-t_0} \le -C_2 e^t$$

for t big enough and $x \in U$. On the other hand, by the normalized condition

$$\int_M e^{-u} dV_g = 1,$$

we can get that

$$u(x(t),t) = \max_x u(x,t) \ge -C.$$

By (5.3), we get that

$$\frac{\partial}{\partial t}(u-\phi) = n - R^T + A \le C.$$

Hence

$$u(x(t),t) - \phi(x(t),t) \le \max_x (u(x,0) - \phi(x,0)) + Ct.$$

It then follows that

$$\max_x \phi(x,t) \ge -C - Ct. \tag{5.6}$$

It is clear that $n + \triangle^T_{g(0)}\phi = n + \triangle_{g(0)}\phi > 0$. Let G_0 be the Green function of $g(0)$ and apply Green's formula to $\phi(x,t)$ with respect to the metric $g(0)$. We can get for $t \ge t_0$ and any x,

$$\begin{aligned}\phi(x,t) &= \int_M \phi dV_0 - \int_M \triangle_{g(0)}(y,t) G_0(x,y) dV_0(y)\\ &\le Vol_0(M\backslash U)\max_x \phi(x,t) + Vol_0(U)\int_U \phi(y,t)dV_0(y) + C\\ &\le Vol_0(M\backslash U)\max_x \phi(x,t) - C_3 e^t + C.\end{aligned}$$

Since $Vol_0(M\backslash U) < Vol_0(M) = 1$, we get that

$$\max_x \phi(x,t) \le -C_4 e^t + C_5. \tag{5.7}$$

Note that all constants C_1, C_2, C_3, C_4, C_5 are independent of $t \ge t_0$. For t big enough, (5.6) contradicts (5.7). Hence there exists a constant $B \ge 1$ such that $u(x,t) \ge -B$ for any (x,t). □

We shall use the notation

$$\Box f = \left(\frac{\partial}{\partial t} - \triangle\right) f.$$

Note that u is basic; standard computations give that

$$\begin{aligned}\Box(\triangle u) &= -|u_{i\bar{j}}|^2 + \triangle u,\\ \Box(|u_i|^2) &= -|u_{ij}|^2 - |u_{i\bar{j}}|^2 + |u_i|^2.\end{aligned} \tag{5.8}$$

Lemma 5.4. *There is a uniform constant C such that*

$$\begin{aligned}|u_i|^2 &\le C(u+2B),\\ R^T &\le C(u+2B).\end{aligned} \tag{5.9}$$

Proof. By Lemma 5.3, there is a uniform constant $B \geq 1$ such that $u(x,t) + B > 0$. Denote

$$H = \frac{|u_i|^2}{u+2B}.$$

Then straightforward computations (see (5.8)) give that

$$\Box H = \frac{-|u_{ij}|^2 - |u_{i\bar{j}}|^2}{u+2B} + \frac{|u_i|^2(2B-A)}{(u+2B)^2} + \frac{2\mathrm{Re}(u_{\bar{j}}(|u_i|^2)_j)}{(u+2B)^2} - \frac{2|u_i|^4}{(u+2B)^3}. \tag{5.10}$$

We compute

$$H_j = \frac{(|u_i|^2)_j}{u+2B} - \frac{|u_i|^2 u_j}{(u+2B)^2}.$$

Hence

$$\frac{\mathrm{Re}(u_{\bar{j}}(|u_i|^2)_j)}{(u+2B)^2} - \frac{|u_i|^4}{(u+2B)^3} = \frac{\mathrm{Re}(u_{\bar{j}}H_j)}{u+2B}. \tag{5.11}$$

It is clear that

$$|u_{\bar{j}}(|u_i|^2)_j| \leq |u_i|^2(|u_{ij}| + |u_{i\bar{j}}|).$$

Then by Cauchy-Schwarz inequality, we compute

$$\frac{|u_{\bar{j}}(|u_i|^2)_j|}{(u+2B)^2} \leq \frac{|u_i|^4}{2(u+2B)^3} + \frac{|u_{ij}|^2 + |u_{i\bar{j}}|^2}{u+2B}. \tag{5.12}$$

Taking (5.10), (5.11) and (5.12) into account, we can choose $\epsilon < 1/4$, such that

$$\Box H \leq \frac{|u_i|^2(2B-A)}{(u+2B)^2} - \frac{\epsilon}{2}\frac{|u_i|^4}{(u+2B)^3} + (2-\epsilon)\frac{\mathrm{Re}(u_{\bar{j}}H_j)}{u+2B}. \tag{5.13}$$

Suppose that

$$\frac{|u_i|^2}{u+2B} \to \infty, \text{ when } t \to \infty.$$

Then there exists time T such that

$$\max_{M\times[0,T]} \frac{|u_i|^2}{u+2B} > 2(2B-A)\epsilon^{-1} \tag{5.14}$$

and the maximum is obtained at some point $(p,T) \in M \times [0,T]$. Since at (p,T),

$$\Box H \geq 0, H_j = 0,$$

it then follows from (5.13) that

$$0 \leq \Box H \leq \frac{|u_i|^2}{(u+2B)^2}\left(2B - A - \frac{\epsilon}{2}\frac{|u_i|^2}{u+2B}\right),$$

which contradicts (5.14). Hence there exists a uniform constant C such that

$$|u_i|^2 \leq C(u+2B).$$

Now we prove that $-\triangle u$ is bounded from above by $C(u+2B)$ for some C. Let

$$K = \frac{-\triangle u}{u+2B}, \quad G = K + \frac{2|u_i|^2}{u+2B}.$$

Straightforward computations give that

$$\Box K = \frac{|u_{i\bar{j}}|^2}{u+2B} + \frac{(-\triangle u)(2B-A)}{(u+2B)^2} + 2\frac{\mathrm{Re}(u_{\bar{j}}K_j)}{u+2B}$$

and

$$\Box G = \frac{-2|u_{ij}|^2 - |u_{i\bar{j}}|^2}{u+2B} + \frac{(-\triangle u + 2|u_i|^2)(2B-A)}{(u+2B)^2} + 2\frac{\mathrm{Re}(u_{\bar{j}})G_j}{u+2B}.$$

Hence at the maximum point of G,

$$0 \le \Box G \le \frac{-|u_{i\bar{j}}|^2}{u+2B} + \frac{-\triangle u(2B-A)}{(u+2B)^2} + \frac{2(2B-A)|u_i|^2}{(u+2B)^2}.$$

We can choose a local coordinate

$$(\triangle u)^2 = (u_{i\bar{i}})^2 \le n u_{i\bar{i}}^2 = n|u_{i\bar{j}}|^2.$$

It then follows that

$$-\frac{|u_{i\bar{j}}|^2}{u+2B} + \frac{-\triangle u(2B-A)}{(u+2B)^2} \le \frac{-\triangle u}{u+2B}\left(\frac{2B-A}{u+2B} - \frac{-\triangle u}{n}\right).$$

By the similar argument (the maximum principle as above), it is clear that

$$\frac{-\triangle u}{u+2B} \le C$$

for some uniform constant C. Since $R^T = n - \triangle u$, the proof is complete. □

In view of Lemma 5.4, we need to bound the Ricci potential from above to bound the scalar curvature and $|\nabla u|$; this bound can be obtained once we can bound the diameter of the manifold along the flow.

Proposition 5.5. *There is a uniform constant C such that*

$$\begin{aligned} u(y,t) &\le C d_t^2(x,y) + C, \\ R^T(y,t) &\le C d_t^2(x,y) + C, \\ |\nabla u| &\le C d_t(x,y) + C, \end{aligned}$$

where d_t is the distance function with respect to $g(t)$ and $u(x,t) = \min_{y\in M} u(y,t)$.

Proof. We assume that $u + B \ge 0$ for some positive constant $B \ge 1$. Note that $|\nabla u|^2 = |u_i|^2$ since u is basic. By Lemma 5.4,

$$|\nabla\sqrt{u+2B}| \le C.$$

Hence

$$\sqrt{u(y,t)+2B} - \sqrt{u(z,t)+2B} \le C d_t(y,z).$$

Note that by the normalization condition,

$$u(x,t) = \min_{y\in M} u(y,t) \le C,$$

where C is a uniform constant.

It then follows that

$$\sqrt{u(y,t)+2B} \le C d_t(y,x) + \sqrt{2B}.$$

Hence there is a uniform constant C such that

$$u(y,t) \le C d_t^2(x,y) + C. \tag{5.15}$$

The other two estimates are direct consequence of Lemma 5.4 and (5.15). □

So far the discussions above are pretty much the same as in the Kähler setting [58]. But we shall see that the Sasakian structure plays a crucial in the following discussion.

6. The Bound on Diameter — Regular or Quasi-regular Case

In this section we consider that (M, ξ, g) is regular or quasi-regular as a Sasakian structure. We shall use the $\mathcal{W}$ functional on Sasaki manifolds to derive some non-collapsing results along the Sasaki-Ricci flow and then prove that the diameter is uniformly bounded along the Sasaki-Ricci flow. This result can be viewed as generalization of Perelman's results on Kähler-Ricci flow to Kähler orbifolds which are underlying Kähler orbifolds of Sasaki manifolds. However the smooth Sasakian structure makes it possible to get these results without detailed discussion of the orbifold singularities on the underlying Kähler orbifolds. In particular, one can allow the singular set to be co-dimension 2. The discussion follows closely the Kähler case [58] except that the $\mathcal{W}$ functional in the Sasaki-Ricci flow involves only basic functions while the distance function of a Sasakian metric is not basic. To overcome this difficulty, we shall explore the relation of the Sasakian structure with its distance function.

When ξ is fixed and the metrics are under deformation generated by basic potentials ϕ such that

$$\eta_\phi = \eta + d_B^c \phi,$$

the distance along the ξ direction does not change under the deformation. Hence we shall introduce the *transverse distance* adapt to the Sasakian structures with ξ fixed when M is regular or quasi-regular. Recall that the Reeb vector field ξ defines a foliation of M through its orbits. The orbits are compact when (M, ξ, g) is regular or quasi-regular. Moreover, M is a principle S^1 bundle (or orbibundle) on a Kähler manifold (or orbifold) Z such that $\pi : (M, g) \to (Z, h)$ is a orbifold Riemannian submersion and

$$g = \pi^* h + \eta \otimes \eta.$$

When (M, ξ, g) is regular (or quasi-regular), the transverse distance is closely related to the distance function on the Kähler manifold (or orbifold) Z, and the results in the Sasaki-Ricci flow on M can be viewed as the corresponding results in the Kähler-Ricci flow on Z, even though we shall not deal with the Kähler-Ricci flow for the orbifold Z directly. Nevertheless we can apply the similar arguments as in the Ricci flow and the Kähler-Ricci flow, for example, see [47, 58, 34] for more details.

Note that the orbit ξ_x is compact for any x. We can define the transverse distance function as

$$d_g^T(x, y) := d(\xi_x, \xi_y), \tag{6.1}$$

where d is the distance function defined by g. Note that d_g^T is not a distance function on M and we shall use $d^T(x, y)$ when there is no confusion. We also define the *transverse diameter* of a Sasaki structure (M, ξ, g) as

$$d_g^T = \max_{x,y\in M} d_g^T(x, y).$$

Proposition 6.1. *Let (M, ξ, g) be a regular or quasi-regular Sasaki structure. For any point $p \in \xi_x$, there exists a point $q \in \xi_y$ such that*

$$d^T(x, y) = d(p, q)$$

In particular,

$$d^T(x, y) = d(x, \xi_y) = d(y, \xi_x). \tag{6.2}$$

As a consequence, the transverse distance function d^T satisfies the triangle inequality,

$$d^T(x,z) \leq d^T(x,y) + d^T(y,z). \tag{6.3}$$

Hence for any $p \in M$, $d^T(p,x)$ is a basic Lipschitz function such that

$$\langle \xi, \nabla d^T(p,x)\rangle = 0.$$

Moreover, let $\pi : (M,g) \to (Z,h)$ be the (orbifold) Riemannian submersion, we have

$$d^T(x,y) = d_h(\pi(x), \pi(y)). \tag{6.4}$$

Proof. Proposition 6.1 is indeed a consequence of results of Reinhart [49] and Molino [43] on Riemannian foliations (dimension 1) and bundle-like metrics ; see, for example, Section 3 (Proposition 3.5, Lemma 3.7, Proposition 3.7) in [43]. Molino proved not only the orbifold structure of foliation space $M/\mathcal{F}_\xi = Z$, but constructed the local (orbifold) coordinates of (Z,h) from some tubular neighborhood of orbits of ξ. With this construction it is not hard to check that (6.4) holds.

We shall give a proof for completeness. Since ξ_x and ξ_y are both compact, there exist $\bar{x} \in \xi_x$, $\bar{y} \in \xi_y$, such that

$$d(\bar{x}, \bar{y}) = d^T(x,y).$$

Now we want to prove that for any $p \in \xi_x$, there exists $q \in \xi_y$ such that $d(p,q) = d^T(x,y)$. Let $\gamma : [0,1] \to M$ be the geodesic such that $\gamma(0) = \bar{x}$, $\gamma(1) = \bar{y}$, and the length of γ is $d(\bar{x}, \bar{y})$. Recall that ξ generates a S^1 action on (M,g) and g is invariant under the action. For any $p \in \xi_x$, there exists an isomorphism $\lambda : (M,g) \to (M,g)$, which is an element in S^1 generated by ξ, such that, $\lambda(\bar{x}) = p$. Let $q = \lambda(\bar{y})$ and $\bar{\gamma} = \lambda \circ \gamma$. Since $\lambda^* g = g$, $\bar{\gamma}$ is also a geodesic which connects p and q such that the length of $\bar{\gamma}$ is the same as the length of γ. It follows that $d(p,q) = d^T(x,y)$. As a direct consequence, (6.2) holds.

For $x, y, z \in M$, we can choose $o \in \xi_x, p \in \xi_y$ and $q \in \xi_z$ such that

$$d(o,p) = d^T(x,y) \text{ and } d(p,q) = d^T(y,z).$$

It then follows that

$$d^T(x,y) + d^T(y,z) = d(o,p) + d(p,q) \geq d(o,q) \geq d^T(x,z). \tag{6.5}$$

In particular, (6.5) together with $d^T(x,y) \leq d(x,y)$ for any $x, y \in M$ imply that d^T is a Lipschitz function on M. For any $p \in M$ fixed, $f(x) = d^T(p,x)$ is constant for x along the geodesics (the orbits) generated by ξ, hence it is a (Lipschitz) basic function.

Now we prove (6.4). Suppose $p \in \xi_x, q \in \xi_y$ such that $d^T(p,q) = d(p,q) = d(\xi_x, \xi_y)$. Let $\gamma(t)$ be one shortest geodesic such that $\gamma(0) = p, \gamma(1) = q$. By the first variation formula of geodesics, we have $\langle \dot{\gamma}(1), \xi\rangle = 0$ at T_qM. By Proposition 1 in [49], $\dot{\gamma}(t) \perp \xi$ for any t. Hence $\gamma(t)$ is an orthogonal geodesic such that the projection $\pi(\gamma(t))$ is also a geodesic in (Z,h), see [43] Proposition 3.5 for example. It is clear that γ only intersects any orbit of ξ at most once, it follows that $\pi(\gamma)$ does not intersect with itself. Moreover the length of γ (with respect to g) is equal to the length of $\pi(\gamma)$ with respect to h. This gives $d^T(p,q) \geq d_h(\pi(p, \pi(q)))$.

Any geodesic in (Z,h) can locally be lifted up to an orthogonal geodesic on (M,g) which has the same length. Suppose for $\pi(p) \neq \pi(q) \in Z$, let $\gamma(t)$ be a shortest geodesic connecting them. We can divide γ into short segments such

that each segment has a lift (an orthogonal geodesic) in (M, g). Note that any orthogonal geodesic in (M, g) can be slid along the orbit of ξ. So we can construct a lift of γ from the local lifts by sliding them. This gives a piece-wise curve in M which connects p and a point $\tilde{q}$ in ξ_q and it has the same length as γ. This gives $d^T(p, q) \leq d_h(\pi(p), \pi(q))$. Hence it completes the proof. $\square$

We can define a *transverse ball* $B_{\xi,g}(x, r)$ as follows,

$$B_{\xi,g}(x, r) = \{y : d^T(x, y) < r\}. \tag{6.6}$$

By Proposition 6.1, we have

$$B_{\xi,g}(x, r) = \{y : d(\xi_x, y) < r\}. \tag{6.7}$$

The following non-collapsing theorem for a transverse ball holds along the Sasaki-Ricci flow, similar to the Ricci flow (Kähler-Ricci flow).

Lemma 6.2. *Let (M, ξ, g_0) be a regular or quasi-regular Sasaki structure and let $g(t)$ be the solution of the Sasaki-Ricci flow with the initial metric g_0. Then there exists a positive constant C such that for every $x \in M$, if $R^T \leq Cr^{-2}$ on $B_{\xi,g(t)}(x, r)$ for $r \in (0, r_0]$, where r_0 is a fixed sufficiently small positive number, then*

$$Vol(B_{\xi,g(t)}(x, r)) \geq Cr^{2n}.$$

Proof. Let $g(t)$ be the solution of Sasaki-Ricci flow. Since R^T is bounded from below, we can assume that r_0 is small enough such that $R^T(x, t) \geq -r_0^{-2}$ for any x, t. We argue by contradiction; suppose the result is not true, then there exists sequences $(p_k, t_k) \in M \times [0, \infty)$ and $t_k \to \infty$ such that $R^T \leq Cr_k^{-2}$ in $B_k = B_{\xi,g(t_k)}(p_k, r_k)$, but $Vol(B_k)r_k^{-2n} \to 0$ as $k \to \infty$. Let Φ be the cut-off function $\Phi : [0, \infty) \to [0, 1]$ such that Φ equals 1 on $[0, 1/2]$, decreases on $[1/2, 1]$ with derivative bounded by 4 and equals 0 on $[1, \infty)$. Denote $\tau_k = r_k^2$ and define

$$w_k = e^{C_k}\Phi(r_k^{-1}d_k^T(p_k, x)),$$

where d_k^T is the transverse distance with respect to time t_k and C_k is the constant such that

$$1 = \int_M w_k^2 \tau_k^{-n} dV_k = e^{2C_k} r_k^{-2n} \int_M \Phi^2(r_k^{-1}d_k^T(p_k, x))dV_k \leq e^{2C_k} r_k^{-2n} Vol(B_k),$$

where we use dV_k to denote the volume element with respect to $g(t_k)$. Hence $C_k \to \infty$ since $Vol(B_k)r_k^{-2n} \to 0$ when $k \to \infty$. We compute (with $g_k = g(t_k)$)

$$\begin{aligned}\mathcal{W}_k &= \tau_k^{-n} \int_M (r_k^2(R^T w_k^2 + 4|\nabla^T w_k|^2) - w_k^2 \log w_k^2)dV_k \\ &\leq e^{2C_k} r_k^{-2n} \int_M (4|\Phi_k^{'}|^2 - \Phi_k^2 \log \Phi_k^2)dV_k + r_k^2 \max_{B_k} R^T - 2C_k,\end{aligned} \tag{6.8}$$

where $\mathcal{W}_k = \mathcal{W}(g_k, w_k, \tau_k)$ and $\Phi_k = \Phi(r_k^{-1}d_k^T(p_k, x))$. Let

$$B_k(r) = B_{\xi,g_k}(p_k, r), \text{ and } V_k(r) = Vol(B_k(r)).$$

For any k fixed,

$$\lim_{r \to 0} \frac{V_k(r)}{V_k(r/2)} = 2^{2n} = 4^n. \tag{6.9}$$

The proof of (6.9) can run as follows. Recall that $\pi : M \to Z$ is a Riemannian submersion over the orbifold Z and M is the S^1 principle orbibundle over Z. For

any $p_k \in M$, $\pi(p_k) \in \pi(B_k(r)) \subset Z$ and $\pi(B_k(r))$ is the geodesic r ball centered at $\pi(p_k)$ with respect to h_k. Hence when r small enough, $B_k(r)$ is a trivial S^1 bundle over the geodesic r ball $B_{h_k}(r)$ of (Z, h_k) centered at $\pi(p_k)$. $\pi(p_k)$ can be either a smooth point or an orbifold singularity in Z. Nevertheless, we have,

$$\lim_{r\to 0} \frac{Vol(B_{h_k}(r))}{Vol(B_{h_k}(r/2))} = 2^{2n} = 4^n.$$

Note that the orbifold singularities in Z is a measure zero set and does not contribute when we compute volume. Without loss of generality, we can assume that $B_{h_k}(r)$ contains only smooth points. Otherwise, we can only consider the set Σ of smooth points in $B_{h_k}(r)$. For generic points (smooth points on Z), any S^1 fibre has the same length. We denote the length of a generic S^1 fibre by l. Note that we have

$$g_k = \pi^* h_k + \eta_k \otimes \eta_k,$$

and $dvol_{g_k} = \pi^*(dvol_{h_k}) \wedge \eta_k$. It follows that

$$Vol_k(B_k(r)) = \int_{B_k(r)} dvol_{g_k} = \int_{B_{h_k}(r)\times S^1} \pi^*(dvol_{h_k}) \wedge \eta_k = l \times Vol(B_{h_k}(r))).$$

This proves (6.9).

We can assume that, in addition,

$$V_k(r_k) \leq 5^n V_k(r_k/2). \tag{6.10}$$

Otherwise, let $r_k^i = 2^{-i} r_k$ for $i \in \mathbb{N}$. By (6.9), we can choose i_0 to be the smallest number such that

$$V_k(r_k^{i_0}) \leq 5^n V_k(r_k^{i_0}/2).$$

Hence for any $i \leq i_0 - 1$,

$$V_k(r_k^i) > 5^n V_k(r_k^i/2) = 5^n V_k(r_k^{i+1}). \tag{6.11}$$

By using (6.11) repeatedly, we can get that

$$\left(r_k^{i_0}\right)^{-2n} V_k(r_k^{i_0}) \leq \left(\frac{4}{5}\right)^{n i_0} r_k^{-2n} V_k(r_k) \to 0$$

when $k \to \infty$. Hence we can replace r_k by $r_k^{i_0}$, which satisfies (6.10) in addition. We then compute, by (6.10),

$$\begin{aligned}\int_M (4|\Phi_k^{'}|^2 - \Phi_k^2 \log \Phi_k^2) dV_k &\leq C(V_k(r_k) - V_k(r_k/2)) \\ &\leq C5^n V_k(r_k/2) \\ &\leq C5^n \int_M \Phi_k^2 dV_k.\end{aligned}$$

Hence we compute, by (6.8) and the above,

$$\mathcal{W}_k \leq C5^n \tau_k^{-n} \int_M w_k^2 dV_k + C - 2C_k \leq C - 2C_k.$$

Note that $\mu(g(t), 1 + (\tau_0 - 1)e^t)$ is non-decreasing function on t. Choose $\tau_0^k = 1 - (1 - r_k^2)e^{-t_k}$, then we get that

$$\mu(g(0), \tau_0^k) \leq \mu(g(t_k), r_k^2) \leq \mathcal{W}_k \to -\infty.$$

Contradiction since $\mu(g(0), \tau)$ is a continuous function of τ in $(0, \infty)$ and $\tau_0^k \to 1$ when $k \to \infty$.

□

Lemma 6.2 is not purely local (it is global in ξ direction); while ξ generates isometries of g, one can further obtain local non-collapsing results and it should be viewed as the corresponding non-collapsing result of the Kähler-Ricci flow on Kähler orbifolds. Now we can bound the diameter of the manifold along the Sasaki-Ricci flow.

Theorem 6.3. *Let (M, ξ, g_0) be a regular or quasi-regular Sasaki structure and let $g(t)$ be the solution of the Sasaki-Ricci flow with the initial metric g_0. Then the transverse diameters $d^T_{g(t)}$ are uniformly bounded. As a consequence there is a uniform constant C such that $diam(M, g(t)) \leq C$.*

Proof. We argue by contradiction. Note that the orbits of ξ are closed geodesics and the length of these closed geodesics does not change along the flow, hence it is uniformly bounded. Suppose the generic orbits have length $l = l(M, \xi, g_0)$. Then for any $x, y \in M$,

$$d_t(x, y) \leq d^T_t(x, y) + l, \tag{6.12}$$

where d_t and d^T_t are the distance function and transverse distance with respect to $g(t)$. Hence we only need to prove that the transverse diameter d^T_t is uniformly bounded along the flow $g(t)$. Let $u(x, t)$ be the transverse Ricci potential defined in (5.1). Recall that $u(x, t)$ is uniformly bounded from below by Lemma 5.3. Choose a point $x_t \in M$ such that

$$u(x_t, t) = \min_{y \in M} u(y, t).$$

Denote

$$d_t(y) = d_t(x_t, y),\ d^T_t(y) = d^T_t(x_t, y).$$

Let $B_\xi(k_1, k_2) = \{y : 2^{k_1} \leq d^T_t(y) \leq 2^{k_2}\}$. Consider the transverse annulus $B_\xi(k, k+1)$ $k \geq 0$. By Proposition 5.5 and (6.12), we have $R^T \leq C 2^{2k}$ on $B_\xi(k, k+1)$. The transverse annulus $B_\xi(k, k+1)$ contains at least 2^{2k-1} transverse ball of radius 2^{-k} which are not intersected with each other. When k is large enough, by Lemma 6.2, we have (with $r = 2^{-k}$)

$$Vol(B_\xi(k, k+1)) \geq \sum_i Vol(B_{\xi, g(t)}(p_i, 2^{-k})) \geq 2^{2k-1} 2^{-kn} C,$$

where $\{p_i\}$ are centers of 2^{2k-1} transverse balls contained in $B_\xi(k, k+1)$.

Claim 6.4. *For every $\epsilon > 0$, there exists k_1, k_2 such that $k_2 - k_1 \gg 1$ such that if d^T_t is large enough, then*

1. *$Vol(B_\xi(k_1, k_2)) < \epsilon$,*
2. *$Vol(B_\xi(k_1, k_2)) \leq 2^{10n} Vol(B_\xi(k_1 + 2, k_2 - 2))$.*
3. *There exists r_1, r_2 and a uniform constant C such that $r_1 \in [k_1, k_1 + 1], r_2 \in [k_2 - 1, k_2]$ and*

$$\int_{B_\xi(r_1, r_2)} R^T dV \leq C Vol(B_\xi(k_1, k_2)).$$

Assume that the diameter of $(M, g(t))$ is not uniformly bounded in t. Hence there exists a sequence $t_i \to \infty$ such that $d^T_{t_i} \to \infty$. Let $\epsilon_i \to 0$ be a sequence of positive numbers. By Claim 6.4, there exist sequences k_1^i, k_2^i such that

$$\begin{aligned} V_i(k_1^i, k_2^i) &:= Vol_{t_i}(B_{\xi,t_i}(k_1^i, k_2^i)) < \epsilon_i, \\ V_i(k_1^i, k_2^i) &\leq 2^{10n} V_i(k_1^i + 2, k_2^i - 2). \end{aligned} \tag{6.13}$$

For each i, we can also find r_1^i and r_2^i as in Claim 6.4. Let Φ_i, for each i, be a cut-off function such that $\Phi_i(t) = 1$ for $t \in [2^{k_1^i+2}, 2^{k_2^i-2}]$ and equals zero for t in $(-\infty, 2^{r_1^i}] \cup [2^{r_2^i}, \infty)$ with the derivative bounded by 2. Define $w_i(y) = e^{C_i}\Phi_i(d^T_{t_i}(x_i, y))$, where $x_i = x_{t_i}$ such that

$$\int_M w_i^2 dV_i = 1.$$

This implies that

$$1 = \int_M w_i^2 dV_i = e^{2C_i} \int_M \Phi_i^2 dV_i \leq e^{2C_i} V_i(k_1^i, k_2^i) = e^{2C_i}\epsilon_i.$$

It follows that $C_i \to \infty$ when $i \to \infty$. We compute

$$\begin{aligned} \mathcal{W}_i &= \mathcal{W}(g(t_i), w_i, 1) \\ &\leq e^{2C_i} \int_M (4|\Phi_i'|^2 - \Phi_i^2 \log \Phi_i^2) dV_i + \int_M R^T w_i^2 dV_i - 2C_i. \end{aligned} \tag{6.14}$$

First of all we have, by Claim 6.4 (see (6.13)),

$$\begin{aligned} \int_M R^T w_i^2 dV_i &\leq e^{2C_i} \int_{B_{\xi,t_i}(r_1^i, r_2^i)} R^T dV_i \\ &\leq e^{2C_i} C V_i(k_1^i, k_2^i) \\ &\leq e^{2C_i} C 2^{10n} V_i(k_1^i + 2, k_2^i - 2) \\ &\leq C 2^{10n} \int_M w_i^2 dV_i \\ &= C 2^{10n}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} e^{2C_i} \int_M (4|\Phi_i'|^2 - \Phi_i^2 \log \Phi_i^2) dV_i &\leq C e^{2C_i} V_i(k_1^i, k_2^i) \\ &\leq e^{2C_i} C 2^{10n} V_i(k_1^i + 2, k_2^i - 2) \\ &\leq C 2^{10n} \int_M w_i^2 dV_i \\ &= C 2^{10n}. \end{aligned}$$

Hence we get

$$\mathcal{W}_i \leq C - 2C_i$$

for some uniform constant C. By the monotonicity of $\mu(g(t), 1)$ (with $\tau_0 \equiv 1$), we get that

$$\mu(g(0), 1) \leq \mu(g(t_i), 1) \leq \mathcal{W}_i \leq C - 2C_i.$$

Contradiction since $C_i \to \infty$. Therefore, there is a uniform bound on the diameter of $(M, g(t))$. ∎

Now we prove Claim 6.4, hence finish the proof of Theorem 6.3.

Proof. Note that the volume of M remains a constant along the flow. If the diameter of $(M, g(t))$ is not uniformly bounded, then for any $\epsilon > 0$, if the diameter of $(M, g(t))$ large enough, there exists K big enough such that for all $k_2 > k_1 \geq K$, $Vol(B_\xi(k_1, k_2)) < \epsilon$. If the estimate (2) does not hold, we have

$$Vol(B_\xi(k_1, k_2)) \geq 2^{10n} Vol(B_\xi(k_1+2, k_2-2)).$$

We then consider $B_\xi(k_1+2, k_2-2)$ instead and ask whether (2) holds or not for $B_\xi(k_1+2, k_2-2)$. We repeat this step if we do not find two numbers such that (1) and (2) are both satisfied. Suppose for every p, at p-th step we cannot find two numbers. Then we would have

$$Vol(B_\xi(k_1, k_2)) \geq 2^{10np} Vol(B_\xi(k_1+2p, k_2-2p)).$$

We can assume that

$$k_1 + 2p \leq k, k_2 - 2p \geq k+1.$$

In particular, we can choose $k \gg 1$ such that

$$k_1 = \frac{k}{2}, k_2 = \frac{3k}{2}, \text{ and } p = \left[\frac{k}{4}\right] - 1.$$

It then follows that

$$\epsilon > Vol(B_\xi(k_1, k_2)) \geq 2^{10np} Vol(B_\xi(k, k+1)) \geq 2^{10np} C 2^{-2nk} 2^{2k-1}.$$

Note that $10p > 2k$ when k is big. This gives a contradiction if k is big enough.

We define a transverse metric sphere $S_\xi(x, r)$ as

$$S_\xi(x, r) = \{y : d^T_t(x, y) = r\}.$$

Hence we have

$$\frac{d}{dr} Vol(B_{\xi, g}(x, r)) = Vol(S_\xi(x, r)).$$

Given $k_1 \ll k_2$ in (1) and (2), there exists $r_1 \in [k_1, k_1+1]$ such that for $r = 2^{r_1}$,

$$Vol(S_\xi(x, r)) \leq \frac{2Vol(B(k_1, k_2))}{2^{k_1}}.$$

Suppose not, we have

$$Vol(B(k_1, k_1+1)) = \int_{2^{k_1}}^{2^{k_1+1}} Vol(S_\xi(x, r)) dr \geq 2Vol(B_\xi(k_1, k_2)).$$

Contradiction since $k_1 \ll k_2$. Similarly there exists $r_2 \in [k_2 - 1, k_2]$ such that

$$Vol(S_\xi(x, 2^{r_2})) \leq \frac{2Vol(B(k_1, k_2))}{2^{k_2}}.$$

We have the estimate,

$$\begin{aligned}
&\int_{B_\xi(r_1, r_2)} R^T - nVol(B_\xi(r_1, r_2)) = \int_{B_\xi(r_1, r_2)} (R^T - n) \\
&= -\int_{B_\xi(r_1, r_2)} \triangle u \\
&\leq \int_{S_\xi(x, 2^{r_1})} |\nabla u| + \int_{S_\xi(x, 2^{r_2})} |\nabla u| \qquad (6.15) \\
&\leq 2CVol(B_\xi(k_1, k_2)) \left(\frac{1}{2^{k_1}} 2^{k_1+1} + \frac{1}{2^{k_2}} 2^{k_2}\right) \\
&\leq CVol(B_\xi(k_1, k_2)),
\end{aligned}$$

where we have used the estimate of $|\nabla u|$ in Proposition 5.5. □

Remark 6.5. *Note that the transverse distance function is Lipschitz, hence the integration by parts in* (6.15) *is justified.*

7. Irregular Sasakian Structure and Approximation

When a Sasakian structure (M, ξ, η, Φ, g) is irregular, Rukimbira [53] proved that one can always approximate an irregular Sasakian structure by a sequence of quasi-regular Sasakian structures $(M, \xi_i, \eta_i, \Phi_i, g_i)$ ([53], see Theorem 7.1.10, [4] also). In particular, one can choose $\xi_i = \xi + \rho_i$ for ρ_i in a commutative Lie algebra spanned by the vector fields defining $\bar{\mathcal{F}}_\xi$, the closure of $\mathcal{F}_\xi$; moreover $\lim \rho_i \to 0$ and η_i, Φ_i, g_i can be expressed in terms of ρ_i; see [4] Theorem 7.1.10 for more details. We can then study the behavior of the Sasaki-Ricci flow for each quasilinear Sasakian metric $(M, \xi_i, \eta_i, K_i, g_i)$. Note that the transverse diameter of $g_i(t)$ is uniformly bounded along the Sasaki-Ricci flow for any t and i large enough, by Theorem 6.3. Since R^T and ∇u are both basic, the approximation above is then enough to bound the transverse scalar curvature R^T and ∇u. Note that the generic orbits for (M, ξ, K, g) are not close, hence the length of the generic orbits of (M, ξ_i, K_i, g_i) do not have a uniform bound when $i \to \infty$. However for a compact Sasakian manifold (or K-contact) manifold, there always exists some close orbits of ξ ([2]). Under the Sasaki-Ricci flow, the length of these closed orbits does not change. Hence we can get some close orbit of ξ on (M, ξ_i, K_i, g_i) under approximation which has a uniformly bounded length. This suffices to prove that the diameter is uniformly bounded along the Sasaki-Ricci flow for irregular Sasaki structure.

The main result in this section is as follows,

Theorem 7.1. *Let (M, ξ, g_0) be a Sasakian structure and let $g(t)$ be the solution of the Sasaki-Ricci flow with the initial metric g_0. There is a uniform constant C such that $R^T_0(t) \leq C, |\nabla u_0(t)| \leq C$ and the diameter is also uniformly bounded.*

To prove this result, first let us recall the following proposition proved in [61].

Proposition 7.2 (Smoczyk-Wang-Zhang, [61])**.** *Let $g(t)$ be the solution of the Sasaki-Ricci flow with initial metric g_0. Then for any $T \in (0, \infty)$, there is a constant C such that*

$$\|\phi(t)\|_{C^k(M,g_0)} \leq C, \|g(t)\|_{C^k} \leq C$$

where C depends on g_0, T, k.

We can prove a stability result of the Sasaki-Ricci flow under approximation using the above estimate.

Proposition 7.3. *Let $(M, \xi_0, \eta_0, \Phi_0, g_0)$ be a Sasakian structure and $g_0(t)$ be the solution of the Sasaki-Ricci flow with initial metric g_0. Suppose $(M, \xi_j, \eta_j, \Phi_j, g_j)$ be a sequence of Sasakian structures such that $(M, \xi_j, \eta_j, \Phi_j, g_j) \to (M, \xi_0, \eta_0, \Phi_0, g_0)$ in C^∞ topology when $j \to \infty$ (we choose a sequence as in the proof of Theorem 7.1.10,* [4]*). Then for any ϵ and $T \in (0, \infty)$, there is a constant $N = N(\epsilon, k, T)$ such that when $j \geq N$, $t \in [0, T]$,*

$$\|g_j(t) - g_0(t)\|_{C^k(M,g_0)} \leq \epsilon.$$

Proof. The statement is a standard stability result for parabolic equations. Suppose $g_k(t)$ is the solution of the Sasaki-Ricci flow for the initial metric g_k, and the corresponding potential $\phi_k(t)$ solves (3.10) with $\phi_k(0) = 0$. We should rewrite (3.10), for each k, as

$$\frac{\partial \phi_k}{\partial t} = \log \frac{\eta_k \wedge (d\eta_k - d \circ \Phi_k \circ d\phi_k)^n}{\eta_k \wedge (d\eta_k)^n} + \phi_k - F_k. \tag{7.1}$$

We can assume that $F_k \to F_0$ in C^∞ topology. Note that, for each k, ξ_k and the transverse complex structure are fixed; hence $d \circ \Phi_k(t) \circ d\phi_k = d \circ \Phi_k \circ d\phi_k = -2\sqrt{-1}\partial_B \bar{\partial}_B \phi_k$. Moreover, by Proposition 7.2, we can assume that $\|\phi_k(t)\|_{C^l} \leq C(l, T, g_0)$ for $t \in [0, T]$. Hence we only need to show that $|\phi_k(t) - \phi_0(t)| \to 0$ when $k \to \infty$ for any t; then by the standard interpolation inequalities we can then get $\|\phi_k(t) - \phi_0(t)\|_{C^l} \to 0$ when $k \to \infty$. To prove $|\phi_k(t) - \phi_0(t)| \to 0$, we consider

$$\frac{\partial}{\partial t}\left(\phi_k(t) - \phi_0(t)\right) = \log \frac{\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_k(t))^n}{\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_0(t))^n} + \phi_k(t) - \phi_0(t) + f_k, \tag{7.2}$$

where

$$f_k = \log \frac{\eta_k \wedge (d\eta_k - d \circ \Phi_k \circ d\phi_k(t))^n}{\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_k(t))^n} - \log \frac{\eta_k \wedge (d\eta_k)^n}{\eta_0 \wedge (d\eta_0)^n} + F_0 - F_k.$$

Note that $\|\phi_k(t)\|_{C^l}$ is uniformly bounded (depending only on T, g_0, l), $f_k \to 0$ when $k \to \infty$. Let $\max |f_k| = c_k$. Let $w(t) = \max_M (\phi_k(t) - \phi_0(t))$. For any $p \in M$ such that $\phi_k(t) - \phi_0(t)$ obtains its maximum at p, then at p,

$$\log \frac{\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_k(t))^n}{\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_0(t))^n} \leq 0. \tag{7.3}$$

We can choose a local coordinate $(x, z_1, \cdots z_n)$ such that at p, $\xi_0 = \partial_x, \eta_0 = dx, d\eta_0 = 2\sqrt{-1} g^T_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$. We compute

$$\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_0(t))^n = 2^n n! (\sqrt{-1})^n \det \left(g^T_{i\bar{j}} + \frac{\partial^2 \phi_0(t)}{\partial z_i \partial z_{\bar{j}}} \right) dx \wedge dZ \wedge d\bar{Z}.$$

Note that $\phi_k(t)$ is not a basic function for ξ_0, but we can compute

$$-d \circ \Phi_0 \circ d\phi_k(t) = 2\sqrt{-1} \frac{\partial^2 \phi_k(t)}{\partial z_i \partial z_{\bar{j}}} dz_i \wedge dz_{\bar{j}} + \theta \wedge dx,$$

where θ is a 1-form. It then follows that

$$\eta_0 \wedge (d\eta_0 - d \circ \Phi_0 \circ d\phi_k(t))^n = 2^n n! (\sqrt{-1})^n \det \left(g^T_{i\bar{j}} + \frac{\partial^2 \phi_k(t)}{\partial z_i \partial z_{\bar{j}}} \right) dx \wedge dZ \wedge d\bar{Z}.$$

Note that the hessian of $\phi_k(t) - \phi_0(t)$ is nonpositive at p, in particular, we have

$$\det \left(g^T_{i\bar{j}} + \frac{\partial^2 \phi_k(t)}{\partial z_i \partial z_{\bar{j}}} \right) \leq \det \left(g^T_{i\bar{j}} + \frac{\partial^2 \phi_0(t)}{\partial z_i \partial z_{\bar{j}}} \right).$$

Hence (7.3) is confirmed. By the standard maximum principle argument and (7.2), we can then get

$$\frac{\partial w(t)}{\partial t} \leq w(t) + c_k.$$

Hence $w(t) \leq c_k(e^T - 1)$. Similarly, we can prove that $-c_k(e^T - 1) \leq \phi_k(t) - \phi_0(t) \leq c_k(e^T - 1)$. □

Now we are in the position to prove Theorem 7.1.

Proof. Pick up a sequence of quasi-regular Sasakian structure (M, ξ_j, g_j) such that $(M, \xi_j, g_j) \to (M, \xi, g_0)$. Then by Theorem 6.3, the transverse diameter $d^T(g_j(t))$ is uniformly bounded for any j and t. By Lemma 5.4, we just need to bound $u_j(t)$, the normalized Ricci potential, from above as in Proposition 5.5. The only difference is that we need to use the fact that u_j is basic and use the transverse diameter to replace the diameter in Proposition 5.5. We can sketch the proof as follows. By Lemma 5.4, we know that

$$|\nabla \sqrt{u_j + 2B}| \le C.$$

Since u_j is basic with respect to (M, ξ_j, g_j), then for any y, z

$$\sqrt{u_j(y,t) + 2B} - \sqrt{u_j(z,t) + 2B} \le C d^T_{g_j(t)}(y, z).$$

Let $u_j(x,t) = \min_M u(y,t)$. Note that $u_j(x,t) \le C$ for some uniformly bounded constant C by the normalized condition (5.2). It implies that

$$\sqrt{u_j(y,t) + 2B} \le C d^T_{g_j(t)}(x, y) + C.$$

This gives a uniform upper bound for $u_j(t)$, hence a uniform upper bound for R^T_j and $|\nabla u_j|$. By Proposition 7.3, for any $t \in (0, \infty)$, $g_j(t) \to g_0(t)$ for j large enough, hence $R^T_j \to R^T_0, |\nabla u_j| \to |\nabla u_0|$.

Now we prove that the diameter is uniformly bounded along the Sasaki-Ricci flow. Recall that there are at least $n+1$ close orbits on a compact K-contact manifold ([54, 55]). Hence we can suppose that (M, ξ, g) has a close orbit $\mathcal{O}$ of ξ. Since $(M, \xi_j, g_j) \to (M, \xi, g_0)$, we can find an orbit $\mathcal{O}_i$ of ξ_i on (M, ξ_i, g_i) whose length converges to the length of $\mathcal{O}$, hence the length $\mathcal{O}_i$ is uniformly bounded. Note that the transverse diameter is also uniformly bounded along the Sasaki-Ricci flow for (M, ξ_i, g_i). This proves that the diameter is also uniformly bounded along the Sasaki flow for (M, ξ_i, g_i). By Proposition 7.3 again, this implies that the diameter is uniformly bounded along the Sasaki-Ricci flow for (M, ξ, g).

□

8. Compact Sasakian Manifolds of Positive Transverse Holomorphic Bisectional Curvature

In this section we consider compact Sasakian manifolds with positive transverse holomorphic bisectional curvature. Transverse holomorphic bisectional curvature can be defined as the holomorphic bisectional curvature for transverse Kähler metric of a Sasakian metric and it was studied in a recent interesting paper [71]. We recall some definitions.

Definition 8.1. *Given two Jinvariant planes σ_1, σ_2 in $\mathcal{D}_x \subset T_x M$, the holomorphic bisectional curvature $H^T(\sigma_1, \sigma_2)$ is defined as*

$$H^T(\sigma_1, \sigma_2) = \langle R^T(X, JX)JY, Y \rangle,$$

where $X \in \sigma_1, Y \in \sigma_2$ are both unit vectors.

It is easy to check that $\langle R^T(X, JX)JY, Y \rangle$ depends only on σ_1, σ_2, hence $H^T(\sigma_1, \sigma_2)$ is well defined. By the first Bianchi identity, one can check that

$$\langle R^T(X, JX)JY, Y \rangle = \langle R^T(Y, X)X, Y \rangle + \langle R^T(Y, JX)JX, Y \rangle.$$

It is often convenient to treat (transverse) holomorphic bisectional curvature in (transverse) holomorphic coordinates. Suppose $u, v \in \mathcal{D}$ are two unit vectors and

let σ_u, σ_v be two J-invariant planes spanned by $\{u, Ju\}$ and $\{v, Jv\}$ respectively. Set

$$U = \frac{1}{2}\left(u - \sqrt{-1}Ju\right), V = \frac{1}{2}\left(v - \sqrt{-1}Jv\right).$$

Denote $R^T(V, \bar{V}; U, \bar{U}) = \langle R^T(V, \bar{U})U, \bar{V}\rangle$, then we have

$$R^T(V, \bar{V}; U, \bar{U}) = \frac{1}{4}H^T(\sigma_u, \sigma_v). \tag{8.1}$$

Definition 8.2. *For $x \in M$, the transverse holomorphic bisectional curvature H^T is positive (or nonnegative) at x if $H^T(\sigma_1, \sigma_2)$ is positive for any two J invariant planes σ_1, σ_2 in $\mathcal{D}_x$. M has positive (nonnegative) transverse holomorphic bisectional curvature if H^T is positive (nonnegative) at any point $x \in M$; equivalently, positivity of transverse holomorphic bisectional curvature means that $R^T(V, \bar{V}; U, \bar{U}) > 0$ for any $V, U \in \mathcal{D}_x \subset T_xM$ and all $x \in M$. For simplicity, we shall also use the notations $R^T \geq 0$ and $R^T > 0$. If a tensor field S has the exact same type as R^T, we can also define its positivity in the same way. If S and T have the same type as R^T, then $S \geq T$ if and only if $S - T \geq 0$.*

It is also useful to consider the transverse holomorphic bisectional curvature locally. Recall that there is an open cover $\{U_\alpha\}$ of M, $V_\alpha \subset \mathbb{C}^n$, submersions $\pi_\alpha : U_\alpha \to V_\alpha$ such that

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

is biholomporphic on $U_\alpha \cap U_\beta$. And the transverse metric g^T is well defined on each U_α. Note that g^T_α is a genuine Kähler metric on V_α and we can identify the transverse holomorphic bisectional curvature of the Sasakian metric g on U_α with the holomorphic bisectional curvature of g^T_α on each V_α. Hence that R^T is positive is equivalent to that R^T_α is positive for all α. Moreover, a Sasaki-Ricci flow solution on M induces a Kähler-Ricci flow solution on V_α for each α. With this relation we can see that the positivity of (transverse) holomorphic bisectional curvature is preserved under the Sasaki-Ricci flow, by extending the results for Kähler-Ricci flow.

For 3-dimensional compact Kähler manifolds, S. Bando [1] proved that the positivity of holomorphic bisectional curvature is preserved along the Kähler-Ricci flow. This was later proved by N. Mok [42] for all dimensions. As a direct consequence of their results, we can get the same property of transverse holomorphic bisectional curvature along the Sasaki-Ricci flow.

We consider the evolution equation of R^T along the Sasaki-Ricci flow, and we denote

$$\frac{\partial R^T}{\partial t} = \triangle^T R^T + F(R^T) + R^T, \tag{8.2}$$

where in the local transverse holomorphic coordinates,

$$\begin{aligned} F(R^T)_{\alpha\bar{\alpha}\beta\bar{\beta}} = \sum_{\mu,\nu} R^T_{\alpha\bar{\alpha}\mu\bar{\nu}} R^T_{\nu\bar{\mu}\beta\bar{\beta}} - \sum_{\mu,\nu} \left|R^T_{\alpha\bar{\mu}\beta\bar{\nu}}\right|^2 + \sum_{\mu,\nu} \left|R^T_{\alpha\bar{\beta}\mu\bar{\nu}}\right|^2 & \\ - \sum_{\mu,\nu} Re\left(R^T_{\alpha\bar{\mu}} R^T_{\mu\bar{\alpha}\beta\bar{\beta}} + R^T_{\beta\bar{\mu}} R^T_{\alpha\bar{\alpha}\mu\bar{\beta}}\right). & \end{aligned} \tag{8.3}$$

Since g^T is a genuine Kähler metric on V_α and it is evolved by the Kähler-Ricci flow, then (8.2) follows from the standard computations in Kähler-Ricci flow.

For any tensor S which has the same type as R^T, we can also define $F(S)$ as in (8.3). As in Kähler case, $F(S)$ satisfies the following property, called the null vector condition.

Proposition 8.3 (Bando; Mok). *If $S \geq 0$ and there exist two nonzero vectors $X, Y \in \mathcal{D}^{1,0}$ such that $S_p(X, \bar{X}; Y, \bar{Y}) = 0$, then $F_p(S)(X, \bar{X}; Y, \bar{Y}) \geq 0$.*

Proof. Note that this is purely local problem and so we can deal with this problem for transverse metric g^T_α on local coordinates V_α. Let $p \in U_\alpha \subset M$ such that $\pi(p) \in V_\alpha$, where U_α and V_α are local coordinates introduced in Section 3. Then g^T_α is evolved by Kähler-Ricci flow on V_α. Hence $F(R^T)$ restricted on V_α is exactly the same as in Kähler case, hence has the formula as in (8.3); see [42] (6) for example. Then the proposition follows from the results of Bando [1] (complex dimension three) and Mok [42] (all dimensions) in Kähler case, see Section 1.2 in [42] for details. □

Then we can get that

Proposition 8.4. *Along the Sasaki-Ricci flow, if the initial metric has nonnegative transverse holomorphic bisectional curvature, then the evolved metrics also have nonnegative transverse holomorphic bisectional curvature; if the initial transverse holomorphic bisectional curvature is positive somewhere, then the transverse holomorphic bisectional curvature is positive for $t > 0$.*

Proof. With Proposition 8.3, the result follows similarly as in Proposition 1 in [1], where Hamilton's maximum principle for tensors was used in Kähler setting. The only slight difference is that the auxiliary function should be basic in Sasaki case. We shall sketch the proof as follows.

It is sufficient to prove the proposition for a short time, thus we consider it in a short closed interval without specification. In particular, all metrics evolved have bounded geometry, namely, they are all equivalent and have bounded curvature (and higher derivatives are bounded also). First we define a parallel tensor field S_0 as in local transverse holomorphic coordinates

$$S_{0 i\bar{j}k\bar{l}} = \frac{1}{2}\left(g^T_{i\bar{j}} g^T_{k\bar{l}} + g^T_{i\bar{l}} g^T_{k\bar{j}}\right),$$

where S_0 has the same type as R^T and $S_0 > 0$ everywhere in the sense of Definition 8.2. Then there exists a positive constant $C \in \mathbb{R}$ such that

$$-CS_0 \leq \frac{dS_0}{dt} \leq CS_0.$$

When f is a basic function, then fS_0 also has the same type as R^T and so $F(R^T + fS_0)$ is well defined. It is clear that $F(S)$ is smooth for S, hence there exists a positive constant D such that

$$F(R^T) \geq F(R^T + fS_0) - D|f|S_0, |f| \leq 1.$$

Now we consider the tensor $R^T+\epsilon f S_0$ for some small positive number ϵ and time dependent function $f(t,x)$. We compute, if f is basic for all t,

$$
\begin{aligned}
\frac{\partial}{\partial t}(R^T+\epsilon_0 f S_0) =& \triangle R^T+F(R^T)+R^T+\epsilon\frac{\partial}{\partial t}S_0+\epsilon f\frac{\partial S_0}{\partial t}\\
=&\triangle(R^T+\epsilon f S_0)+F(R^T+\epsilon f S_0)+R^T+\epsilon f S_0\\
&+\epsilon\left(\frac{\partial f}{\partial t}-\triangle f\right)S_0+F(R^T)-F(R^T+\epsilon f S_0)\\
&+\epsilon f\left(\frac{\partial S_0}{\partial t}-S_0\right)\\
\geq&\triangle(R^T+\epsilon f S_0)+F(R^T+\epsilon f S_0)+R^T+\epsilon f S_0\\
&+\epsilon S_0\left(\frac{\partial f}{\partial t}-\triangle f-(C+D+1)f\right).
\end{aligned}
\tag{8.4}
$$

Let $f(0,\cdot)\equiv 1$ and let $f(t,x)$ satisfy the equation

$$\frac{\partial f}{\partial t}-\triangle f-(C+D+1)f=1.$$

In local coordinates, we can write

$$\triangle f=\xi^2 f+g_T^{i\bar{j}}\partial_i\partial_{\bar{j}}f.$$

Clearly, $\xi g^T_{i\bar{j}}=0$ and ξ can commute with $\partial_i,\partial_{\bar{j}}$ when taking derivatives. So we have

$$\xi(\triangle f)=\triangle(\xi f).$$

Hence ξf satisfy the equation

$$\frac{\partial(\xi f)}{\partial t}=\triangle(\xi f)-(C+D+1)\xi f.$$

By the maximum principle, $\xi f\equiv 0$ since $\xi f=0$ at $t=0$. Hence (8.4) is justified. Moreover it is clear $f>0$ for $t>0$. Since $R^T\geq 0$ at $t=0$, $R^T+\epsilon f S_0>0$ at $t=0$. Now we claim $R^T+\epsilon f S_0>0$ for all t and small ϵ. If not, there is a first time t_0 such that $R^T+\epsilon f S_0(X,\bar{X},Y,\bar{Y})=0$ at some point (t_0,p) for some nonzero vectors $X,Y\in\mathcal{D}^{1,0}$ and $R^T+\epsilon f S_0>0$ for $t<t_0$. We now consider the problem locally on V_α and let $X_\alpha=\pi_*(X),Y_\alpha=\pi_*(Y)\in T^{1,0}V_\alpha$. We can extend X_α,Y_α as follows. At $t=t_0$, we extend X_α,Y_α to a normal neighborhood of $(t_0,\pi(p))$ in $[0,t_0]\times V_\alpha$ by parallel transformation along radial geodesics of g^T_α at $(t_0,\pi(p))$ such that $\nabla X_\alpha=\nabla Y_\alpha=0$ at $(t_0,\pi(p))$, and then we extend X_α,Y_α around t_0 such that $\partial_t X_\alpha,\partial_t Y_\alpha=0$. We can then compute, at (t_0,p),

$$\frac{\partial}{\partial t}(R^T+\epsilon f S_0)(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)=\left(\frac{\partial}{\partial t}(R^T+\epsilon f S_0)\right)(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)\leq 0,$$

and

$$0\leq\triangle\left((R^T+\epsilon f S_0)(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)\right)=(\triangle(R^T+\epsilon f S_0))(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha).$$

But by (8.4), we get

$$
\begin{aligned}
0&\geq\left(\frac{\partial}{\partial t}(R^T+\epsilon f S_0)\right)(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)\\
&\geq(\triangle(R^T+\epsilon f S_0))(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)+\epsilon S_0(X_\alpha,\bar{X}_\alpha,Y_\alpha,\bar{Y}_\alpha)>0,
\end{aligned}
$$

since $F(R^T+\epsilon f S_0)$ satisfies the null vector condition and $(R^T+\epsilon f S_0)(X,\bar{X},Y,\bar{Y}) = 0$. Contradiction. Now let $\epsilon \to 0$, we prove that $R^T \geq 0$ for all t.

The proof for the last statement of the proposition is also similar. But we shall choose the function f more carefully. Recall $R^T \geq 0$ and R^T is positive somewhere, at $t = 0$. We can define a function $f_0(p) = \min R^T_p(X,\bar{X},Y,\bar{Y})$ for $p \in M, X, Y \in \mathcal{D}^{1,0}_p, |X| = |Y| = 1$. It is clear f_0 is nonnegative and cannot be identically zero. Since ξ acts on g isometrically, f_0 is constant along any orbit of ξ. Hence $\xi f_0 = 0$ is well defined. Similarly we compute

$$\frac{\partial}{\partial t}(R^T - fS_0) \geq \triangle(R^T - fS_0) + F(R^T - fS_0) + \left(-\frac{\partial f}{\partial t} + \triangle f - (C+D)|f|\right) S_0.$$

We choose $f(0,\cdot) = f_0$ such that

$$\frac{\partial f}{\partial t} = \triangle f - (C+D)f.$$

Note that $\xi f \equiv 0$ since $\xi f_0 = 0$. By our choice of f, $R^T - fS_0 \geq 0$ at $t = 0$ and by the similar argument as above, we can get that $R^T - fS_0 \geq 0$. If we let $\tilde{f} = e^{-(C+D)t} f$, then we can get that

$$\frac{\partial \tilde{f}}{\partial t} = \triangle \tilde{f}.$$

We then get $\tilde{f} > 0$ for $t > 0$, hence $f(t) > 0$ when $t > 0$. It follows that $R^T > 0$ for $t > 0$. This completes the proof. □

Then we study the Sasaki-Ricci flow for the metric with transverse holomorphic bisectional curvature. We have

Theorem 8.5. *Suppose (M,ξ,g) is a Sasakian structure such that g has nonnegative transverse holomorphic bisectional curvature then the Sasaki-Ricci flow with initial metric g exists for all time with bounded curvature and it converges to a Sasaki-Ricci soliton (M,ξ_∞,g_∞) subsequently.*

Proof. Since the transverse holomorphic bisection curvature becomes positive along the Sasaki-Ricci flow, then the transverse holomorphic bisectional curvature is then bounded by its transverse scalar curvature, which is bounded along the flow by Theorem 7.1. Hence the transverse sectional curvature defined by g^T is bounded. It follows that the sectional curvature of g for any two unit vectors $X, Y \in \mathcal{D}$ is then bounded by (3.5). Moreover, by the definition, the sectional curvature of g for any two plane in TM containing ξ is 1, hence the sectional curvature of g is uniformly bounded along the Sasaki-Ricci flow. Moreover the diameter of M is uniformly bounded and the volume is fixed along the flow. It is well known that the Sobolev constants and the injectivity radii are uniformly bounded along the flow. Hence for any $t_i \to \infty$, there is a subsequence such that $(M, g(t_i))$ converges to a compact manifold (M,ξ_∞,g_∞) in Cheeger-Gromov sense. Namely there is a sequence of diffeomorphisms Ψ_i such that $\Psi_i^* g(t_i)$ converges to g_∞, if necessarily, after passing to a subsequence. And $\Psi_{i*}\xi$ and $\Psi_i^*\eta(t_i)$ converge to ξ_∞, η_∞ as tensor fields, which are compatible with g_∞ such that $(M,\xi_\infty,g_\infty,\eta_\infty)$ defines a Sasakian structure.

To understand the structure of the limit metric, we recall Hamilton's compactness theorem for Ricci flow, which applies to the Sasaki setting provided that the curvature, the diameter and the volume are all uniformly bounded. For any $t_i \to \infty$,

we consider the sequence of Sasakian-Ricci flow $(M, g(t_i + t))$. Then after passing to some subsequence, it converges to the limit Sasaki-Ricci flow $(M, g_\infty(t))$. As in Kähler setting [57], one can prove that the limit flow is actually one parameter family of Sasakian-Ricci solitons evolved along the Sasaki-Ricci flow. First observe that along the flow, the μ functional is uniformly bounded from above. Let $u(t)$ be the normalized (transverse) Ricci potential, then

$$\mu(g(t), 1) \leq \int_M e^{-u}(R^T + u + |\nabla u|^2) dV_{g(t)} \leq C$$

for some uniformly bounded constant C since $R^T, |u|$ and $|\nabla u|$ are uniformly bounded along the flow. And $\mu(g(t), 1)$ is increasing, hence $\lim_{t\to\infty} \mu(g(t), 1)$ exists. This implies that the μ functional for $(M, \xi_\infty, g_\infty(t))$ is a constant. To show $g_\infty(t)$ is a Sasakian-Ricci soliton, first we assume that for some $T \in (0, \infty)$, $\mu(g_\infty(T), 1)$ has a smooth positive minimizer w_0 (we are using the form of (4.28)), then consider the backward heat equation for w_0 such that

$$\frac{dw(t)}{dt} = -\triangle w + (R^T - n)w, w(T) = w_0.$$

As in Proposition 4.8, $w(t)$ is smooth and positive. In particular, $\mathcal{W}(g_\infty(t), w(t), 1)$ is increasing for $t \in [0, T]$. Hence we get $\mu(g_\infty(t), 1) \leq \mathcal{W}(g_\infty(t), w(t), 1) \leq \mu(g_\infty(T), 1)$. But $\mu(g_\infty(t), 1)$ is constant along the flow. This implies that $w(t)$ minimizes $\mathcal{W}(g_\infty, w(t), 1)$. In particular, let $f(t) = -log w(t)$, then we have

$$\frac{d}{dt}\mathcal{W}(g_\infty(t), f(t), 1) \equiv 0 \tag{8.5}$$

for $t \in [0, T]$. By Proposition 4.4, we get that, for any $t \in [0, T]$,

$$R^T_{\infty i\bar{j}} + f_{i\bar{j}} - g^T_{\infty i\bar{j}} = 0, f_{ij} = 0.$$

In general, let w_0 be a nonnegative minimizer in $W^{1,2}_B$ for $\mu(g_\infty(T), 1)$. Let $w(t)$ be the solution of the backward heat equation

$$\frac{\partial w}{\partial t} = -\triangle w + (R^T - n\tau^{-1})w$$

such that $w(T) = w_0$. The standard parabolic regularity theory implies that $w(t) \in C^\infty$. By Proposition 4.8, $w(t) \geq 0$ for $t \in [0, T)$. It is clear that $w(t)$ can never be identically zero, hence $w(t) > 0$ for all $t \in [0, T)$. Note we can repeat the argument above to get that $w(t)$ minimizes $\mu(g_\infty(t), 1)$ and that $(M, g_\infty(t))$ is a Sasaki-Ricci soliton for any $t \in [0, T)$. Hence $(M, g_\infty(t))$ is a Sasaki-Ricci soliton for any $t \in [0, \infty)$. □

Remark 8.6. *On Riemannian surfaces, there exists nontrivial Ricci soliton with positive curvature on sphere with orbifold singularity* [68, 17]. *Note that this corresponds to the case of quasi-regular Sasakian metric on wighted 3 sphere with positive (transverse) bisectional curvature, in particular the limit soliton does not have to be transverse Kahler-Einstein. It would be very interesting to classify Sasaki-Ricci soliton with positive transverse bisectional curvature. I am grateful to Xiuxiong Chen and Song Sun for valuable discussions on this topic.*

9. Appendix

We summarize some results for Sakakian manifolds with the positive transverse bisectional curvature. The topology of compact Sasakian manifolds with positive curvature in suitable sense are well studied, for example see [27, 46, 63, 64, 5].

First we need several formulas about basic forms on Sasakian manifolds. All these formulas are similar as as the corresponding formulas in the Kähler setting, see Morrow-Kodaira [45] for the corresponding Kähler version.

Proposition 9.1. *Let $\phi = \phi_{A\bar{B}} dz_A \wedge dz_{\bar{B}}$ be a basic (p, q) form on a Sasakian manifold (M, ξ, g). Then*

$$\partial\phi = \nabla_i^T \phi_{A\bar{B}} dz_i \wedge dz_A \wedge dz_{\bar{B}},$$
$$\bar{\partial}\phi = \nabla_{\bar{j}}^T \phi_{A\bar{B}} dz_{\bar{j}} \wedge dz_A \wedge dz_{\bar{B}}.$$

Proof. We have

$$\nabla_i^T \phi_{A\bar{B}} = \partial_i \phi_{A\bar{B}} - \Gamma^\sigma_{i\alpha_k} \phi_{\alpha_1 \cdots \alpha_{k-1}\sigma\alpha_{k+1}\cdots\alpha_p \bar{B}}$$

Note that $\Gamma^\sigma_{i\alpha_k} = \Gamma^\sigma_{\alpha_k i}$ (Γ is symmetric on i and α_k), while $dz_i \wedge dz_A \wedge dz_{\bar{B}}$ is skew-symmetric on dz_i and dz_{α_k}. Hence

$$\nabla_i^T \phi_{A\bar{B}} dz_i \wedge dz_A \wedge dz_{\bar{B}} = \partial_i \phi_{A\bar{B}} dz_i \wedge dz_A \wedge dz_{\bar{B}} = \partial\phi.$$

Similarly we can get the formula for $\bar{\partial}\phi$. □

Proposition 9.2. *Let (M, ξ, g) be a compact Sasakian manifold. Then*

$$\begin{aligned} &(\bar{\partial}_B \phi, \psi) = (\phi, \bar{\partial}_B^* \psi), \textit{for } \phi \in \Omega_B^{p,q-1}, \psi \in \Omega_B^{p,q}, \\ &(\partial_B \phi, \psi) = (\phi, \partial_B^* \psi), \textit{for } \phi \in \Omega_B^{p-1,q}, \psi \in \Omega_B^{p,q}, \\ &(d_B \phi, \psi) = (\phi, \delta_B \psi), \textit{for } \phi \in \Omega_B^{r-1}, \psi \in \Omega_B^{r}. \end{aligned} \tag{9.1}$$

In particular, we have

$$\bar{\partial}_B^* \psi_{A\bar{B}} = -(-1)^p g_T^{i\bar{j}} \nabla_i \psi_{A\bar{j}\bar{B}},$$

where $\psi \in \Omega_B^{p,q+1}$.

Proof. The formulas in (9.1) are well known, see [33] for example (the authors deal with only real forms in [33]. But it is a straightforward extension to complex forms). For the sake of completeness, we include a proof of the first identity in (9.1), see [45] for the Kähler setting for example. By definition

$$(\bar{\partial}_B \phi, \psi) = \int_M \bar{\partial}_B \phi \wedge *_B \bar{\psi} \wedge \eta.$$

Let $\Phi = \phi \wedge *_B \bar{\psi} \in \Omega_B^{n,n-1}$. Hence $\partial_B \Phi = 0$ and $d\Phi = d_B \Phi = \bar{\partial}_B \Phi$. Note that $d\eta = \sqrt{-1} g^T_{i\bar{j}} dz_i \wedge dz_{\bar{j}} \in \Omega_B^{1,1}$. We have

$$d(\Phi \wedge \eta) = -\Phi \wedge d\eta + d_B \Phi \wedge \eta = \bar{\partial}_B \Phi \wedge \eta.$$

It follows that

$$0 = \int_M \bar{\partial}_B \Phi \wedge \eta = \int_M \bar{\partial}_B \phi \wedge *_B \bar{\psi} \wedge \eta + (-1)^{p+q-1} \int_M \phi \wedge \bar{\partial}_B *_B \bar{\psi} \wedge \eta.$$

Hence

$$\begin{aligned}(\bar\partial_B\phi,\psi) &= (-1)^{p+q}\int_M \phi\wedge\bar\partial_B *_B \bar\psi\wedge\eta\\ &= (-1)^{p+q}\int_M \phi\wedge(-1)^{2n+1-p-q} *_B *_B\bar\partial_B *_B \bar\psi\wedge\eta\\ &= \int_M \phi\wedge *_B\overline{(-1) *_B \partial_B *_B \psi}\wedge\eta\\ &= (\phi,\bar\partial^*\psi).\end{aligned}$$

Note that $*_B$ can be characterized by

$$\phi\wedge *_B\bar\phi = \frac{(d\eta)^n}{2^n n!} = \det(g^T_{i\bar j})dZ\wedge d\bar Z$$

for basic forms. Then we have, for basic (p,q) forms $\phi = \phi_{A\bar B}dz_A\wedge dz_{\bar B}$ and $\psi = \psi_{C\bar D}dz_C\wedge dz_{\bar D}$,

$$(\phi,\psi) = \int_M \phi_{A\bar B}\overline{\psi_{C\bar D}}g_T^{A\bar C}g_T^{B\bar D}dvol_g. \tag{9.2}$$

To compute $\bar\partial^*_B\psi$, let $\phi = \phi_{A\bar B}dz_A\wedge dz_{\bar B}$ be a basic (p,q) form and $\psi = \psi_{C\bar D_0}dz_C\wedge dz_{\bar D_0}$ be a basic $(p,q+1)$ form. We denote $D_0 = iD$ and $B_0 = jB$. Then we have,

$$\begin{aligned}(\bar\partial_B\phi,\psi) &= \int_M (-1)^p\nabla^T_{\bar j}\phi_{A\bar B}\overline{\psi_{C\bar D_0}}g_T^{A\bar C}g_T^{D_0\bar B_0}dvol_g\\ &= -(-1)^p\int_M \phi_{A\bar B}\overline{\nabla^T_{\bar j}\psi_{C\bar i\bar D}}g_T^{A\bar C}g_T^{D\bar B}g_T^{i\bar j}dvol_g\\ &= (\phi,\bar\partial^*_B\psi).\end{aligned}$$

Using (9.2) and the above, we can get that

$$\bar\partial^*_B\psi_{C\bar D} = -(-1)^p g_T^{j\bar i}\nabla_j\psi_{C\bar i\bar D}.$$

□

Then we have the following,

Proposition 9.3. *Let $\Phi = \Phi_{A\bar B}dz_A\wedge dz_{\bar B}$ be a basic (p,q) form. Then we have*

$$\triangle_{\bar\partial}\Phi_{A\bar B} = -g_T^{i\bar j}\nabla^T_i\nabla^T_{\bar j}\Phi_{A\bar B} - \sum_k(-1)^k g_T^{i\bar j}[\nabla^T_i,\nabla^T_{\bar\beta_k}]\Phi_{A\bar j\bar\beta_1\cdots\hat{\bar\beta}_k\cdots\bar\beta_q}. \tag{9.3}$$

Moreover, when $\Phi = \Phi_{\bar B}dz_{\bar B}$ is a basic $(0,q)$ form, we have

$$\triangle_{\bar\partial}\Phi_{\bar B} = -g_T^{i\bar j}\nabla^T_i\nabla^T_{\bar j}\Phi_{\bar B} + \sum_k R^T_{\tau\bar\beta_k}\Phi_{\bar\beta_1\cdots\bar\beta_{k-1}\bar\tau\bar\beta_{k+1}\cdots\bar\beta_q} \tag{9.4}$$

Proof. Recall $\triangle_{\bar\partial} = \bar\partial_B\bar\partial^*_B + \bar\partial^*_B\bar\partial_B$. (9.3) follows from the direct computation with Proposition 9.1 and 9.2 and (9.4) is a direct consequence of (9.3). Note that we can consider this problem locally involved with transverse Kähler metric g^T_α on V_α. By Proposition 9.1 and 9.2, $\bar\partial_B, \bar\partial^*_B$ can be viewed as the corresponding operator of the Kähler metric g^T_α on V_α. Hence the computation of $\triangle_{\bar\partial}$ on basic forms has the same local formula as the corresponding formula in the Kähler setting, with the Kähler metric replaced by the transverse Kähler metric (see [45] Chapter 2, Theorem 6.1 for the Kähler setting; note that there is a sign difference in [45] for $R_{i\bar j}$ with ours). □

Proposition 9.4. *Let (M, g) be a compact Sasakian manifold of dimension $(2n+1)$ such that $Ric^T \geq 0$ and Ric^T is positive at one point. Then*

$$b_1(M) = b_1^B = 0.$$

Moreover, we have the following transverse vanishing theorem

$$H^{q,0}(\mathcal{F}_\xi) = H^{0,q}(\mathcal{F}_\xi) = 0, q = 1, \cdots, n.$$

Proof. The statement $b_1(M) = 0$ is proved by Tanno with the assumption $Ric + 2g > 0$ (see [64] Theorem 3.4). Given that $Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y)$ for $X, Y \in \mathcal{D}$, $Ric(\xi, \xi) = 2n$, $Ric(\xi, X) = 0$, $X \in \mathcal{D}$, then $Ric^T > 0$ is equivalent to $Ric + 2g > 0$. When (M, g) is assumed to be positive Sasakian structure, Boyer-Galicki-Nakamaye [5] proved $b_1 = 0$, and also proved the vanishing theorem when (M, g) is quasi-regular. We would use the transverse Hodge theory to prove that $H^1_B(\mathcal{F}_\xi) = 0$ and we know that $H^1(M, \mathbb{R}) \approx H^1_B(\mathcal{F}_\xi)$ (see [4], page 215 for example). Suppose $w = w_{\bar{l}} dz_{\bar{l}}$ is a basic harmonic $(0, 1)$ form. Then by Proposition 9.3 we have

$$\triangle_{\bar{\partial}} w_{\bar{l}} = -g_T^{i\bar{j}} \nabla_i^T \nabla_{\bar{j}}^T w_{\bar{l}} + R^T_{p\bar{l}} w_{\bar{p}} = -g_T^{i\bar{j}} \nabla_{\bar{j}}^T \nabla_i^T w_{\bar{l}} = 0.$$

It then follows that

$$\int_M R^T_{p\bar{l}} w_{\bar{p}} \bar{w}_{\bar{l}} - g_T^{i\bar{j}} \nabla_i^T \nabla_{\bar{j}}^T w_{\bar{l}} \bar{w}_{\bar{l}} = -\int_M g_T^{i\bar{j}} \nabla_{\bar{j}}^T \nabla_i^T w_{\bar{l}} \bar{w}_{\bar{l}} = 0.$$

Hence

$$\int_M R^T_{p\bar{l}} w_{\bar{p}} \bar{w}_{\bar{l}} + |\nabla_{\bar{j}}^T w_{\bar{l}}|^2 = \int_M |\nabla_i^T w_{\bar{l}}|^2 = 0.$$

This implies that $\nabla^T w_{\bar{l}} = 0$ and $R^T_{p\bar{l}} w_{\bar{p}} \bar{w}_{\bar{l}} = 0$. Since $Ric^T \geq 0$ and it is positive at some point p, it follows that $w_{\bar{l}}(p) = 0$. Hence $w_{\bar{l}} \equiv 0$. It follows that $b_B^{0,1} = b_B^{1,0} = 0$. Hence $b_B^1 = 0$. Now we prove $b_B^{q,0} = b_B^{0,q}$ $q > 1$ in the similar way. Suppose $\Phi = \Phi_{\bar{B}} dz_{\bar{B}}$ is a harmonic $(0, q)$ form. Then by Proposition 9.3 we have,

$$\triangle_{\bar{\partial}} \Phi_{\bar{B}} = -g_T^{i\bar{j}} \nabla_i^T \nabla_{\bar{j}}^T \Phi_{\bar{B}} + \sum_k R^T_{\tau\bar{\beta}_k} \Phi_{\bar{\beta}_1 \cdots \bar{\beta}_{k-1} \bar{\tau} \bar{\beta}_{k+1} \cdots \bar{\beta}_q} = -g_T^{i\bar{j}} \nabla_{\bar{j}}^T \nabla_i^T \Phi_{\bar{B}} = 0.$$

It then follows that

$$\int_M R_{\tau\bar{\beta}_k} \Phi_{\bar{\beta}_1 \cdots \bar{\beta}_{k-1} \bar{\tau} \bar{\beta}_{k+1} \cdots \bar{\beta}_q} \overline{\Phi_{\bar{\beta}_1 \cdots \bar{\beta}_q}} + |\nabla_{\bar{j}}^T \Phi_{\bar{B}}|^2 = \int_M |\nabla_i^T \Phi_{\bar{B}}|^2 = 0$$

By the positivity assumption on Ric^T, we can get that $\Phi_{\bar{B}} \equiv 0$. □

Proposition 9.5. *Let (M, g) be a compact Sasakian manifold of dimension $(2n+1)$ such that the transverse bisectional curvature $R^T \geq 0$ and it is positive at one point. Then*

$$b_2(M) = 0, b_2^B(\mathcal{F}_\xi) = 1.$$

Proof. First we have $b_1(M) = b_1^B = 0$. Note that we have the following exact sequence (see [4], page 215 for example)

$$0 = H^1(M, \mathbb{R}) \to \mathbb{R} \to H^2_B(\mathcal{F}_\xi) \to H^2(M, \mathbb{R}) \to H^1_B(\mathcal{F}_\xi) = 0.$$

It follows that $b_2(M) = b_2^B - 1$. Now we shall use the transverse Hodge theory to show that $b_2^B = 1$ if the transverse bisectional curvature is positive. By Proposition 9.4, we have $b_B^{2,0} = b_B^{0,2} = 0$. Now we prove $b_B^{1,1} = 1$ (see [28] for the Kähler setting).

Let $\psi = \psi_{k\bar{l}} dz_k \wedge dz_{\bar{l}}$ be a basic harmonic $(1,1)$ form. Then by Proposition 9.3 we have

$$\begin{aligned}\triangle_{\bar{\partial}} \psi_{k\bar{l}} &= -g_T^{i\bar{j}} \nabla_i^T \nabla_{\bar{j}}^T \psi_{k\bar{l}} - g_T^{i\bar{j}} R^T_{i\bar{q}k\bar{l}} \psi_{q\bar{j}} + R^T_{p\bar{l}} \psi_{k\bar{p}} \\ &= -g_T^{i\bar{j}} \nabla_{\bar{j}}^T \nabla_i^T \psi_{k\bar{l}} - g_T^{i\bar{j}} R^T_{i\bar{q}k\bar{l}} \psi_{q\bar{j}} + 2R^T_{p\bar{l}} \phi_{k\bar{p}} - R^T_{k\bar{p}} \psi_{p\bar{l}}\end{aligned}$$

Choose an orthonormal frame such that $\psi_{k\bar{l}} = \delta_{kl} \psi_{k\bar{k}}$, then we get

$$\left(-g_T^{i\bar{j}} R^T_{i\bar{q}k\bar{l}} \psi_{q\bar{j}} + R^T_{p\bar{l}} \psi_{k\bar{p}}\right) \overline{\psi_{k\bar{l}}} = \sum_{i<j} R^T_{i\bar{i}j\bar{j}} (\psi_{i\bar{i}} - \psi_{j\bar{j}})^2 \geq 0,$$

and

$$\left(-g_T^{i\bar{j}} R^T_{i\bar{q}k\bar{l}} \psi_{q\bar{j}} + 2R^T_{p\bar{l}} \phi_{k\bar{p}} - R^T_{k\bar{p}} \psi_{p\bar{l}}\right) \overline{\psi_{k\bar{l}}} = \sum_{i<j} R^T_{i\bar{i}j\bar{j}} (\psi_{i\bar{i}} - \psi_{j\bar{j}})^2 \geq 0.$$

By the positivity assumption on $R^T_{i\bar{j}k\bar{l}}$, it then follows that $\nabla^T \psi \equiv 0$ and $\psi_{i\bar{i}} = \psi_{j\bar{j}}$ at some point p for all i, j. Hence $\psi = \lambda d\eta$ for some constant λ. Hence $b_B^{1,1} = 1$. □

If (M, g) is Sasaki-Einstein and (M, g) has positive transverse bisectional curvature, then (M, g) has to be a space form with positive curvature 1.

Proposition 9.6. *If (M, g) is a Sasaki-Einstein metric such that the transverse Kähler structure has positive holomorphic bisectional curvature, then (M, g) has constant sectional curvature* 1.

Proof. When (M, g) is a Kähler manifold such that the bisectional curvature is positive and g is Kähler-Einstein, then Berger [3] and Goldberg-Kobayashi [28] proved that g has constant bisectional curvature. In the Sasakian setting, if (M, g) is Sasaki-Einstein with positive sectional curvature, then (M, g) has constant sectional curvature 1 (c.f Moskal [46] and [4] Chapter 11).

Here we follow Chen-Tian's argument [16] in Kähler setting using the maximum principle to prove that the transverse Kähler metric g^T has constant transverse bisectional curvature. One can easily get that the sectional curvature of g is 1 for a Sasaki-Einstein metric if its transverse Kähler metric g^T has constant transverse bisectional curvature,

$$R^T_{i\bar{j}k\bar{l}} = 2(g^T_{i\bar{j}} g^T_{k\bar{l}} + g^T_{i\bar{l}} g^T_{k\bar{j}}).$$

Suppose that g is a Sasaki-Einstein metric and g^T is transverse Kähler Einstein with positive transverse bisectional curvature. Then we can apply the argument of the maximum principle as in [16] (Lemma 8.19) to the Sasakian case to show that g^T has constant transverse bisectional curvature. The compactness of M is only needed to find a maximum (or minimum) of auxiliary functions. After that we can do all computations locally on some V_α. Only the transverse Kähler structure is involved. Hence the argument in [16] is applicable and the details to the Sasakian case can be carried out as in Proposition 8.4. We shall skip the details. It is clear that if we only assume that the transverse bisectional curvature is nonnegative and it is positive at one point, we can get the same conclusion.

□

With the result in [71], one can actually prove

Proposition 9.7. *If (M, g) is a Sasakian metric with constant scalar curvature and positive transverse bisectional curvature, then (M, g) has constant transverse*

bisectional curvature. In particular, there is a D-homothetic transformation such that $\tilde{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta$ has constant sectional curvature 1 *for some* $\alpha > 0$.

Proof. In [71], X. Zhang proved that (M, g) is transverse Kähler-Einstein. We can actually give a simple proof of this fact as follows. Since g has positive transverse bisectional curvature, by Proposition 9.5, $b_2^B(\mathcal{F}_\xi) = 1$. If the transverse scalar curvature R^T is constant, we have

$$\sqrt{-1}\bar{\partial}^*\rho^T = -[\Lambda, \partial]\rho^T = \partial(\Lambda\rho^T) = \partial R^T = 0,$$

hence the transverse Ricci form ρ^T is a harmonic form, and it is a basic $(1,1)$ form. If $b_2^B(\mathcal{F}_\xi) = 1$, then $\rho^T = R^T d\eta/2$, hence it is a transverse Kähler metric with constant transverse Ricci curvature. As in Proposition 9.6, one can further show that g^T has constant transverse bisectional curvature. Since $R^T > 0$ and g^T is transverse Kähler-Einstein, it is well known that after a D-homothetic transformation such that $\tilde{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta$ foe some $\alpha > 0$, $\tilde{g}$ is Sasakian-Einstein. Since $\tilde{g}^T$ is just a rescaling of g^T, it has constant transverse bisectional curvature. Hence by Proposition 9.6, $\tilde{g}$ has constant sectional curvature 1. It is clear that if we only assume that the transverse bisectional curvature is nonnegative and it is positive at one point, we can get the same conclusion. □

We then discuss the minimizers of $\mathcal{W}$ functional as in (4.28). We finish the proof of Theorem 4.6.

Proof. The existence of a nonnegative minimizer is almost identical as in [52], Section 1. To prove actually w_0 satisfies (4.30), we follow [51, 52] with a slight modification. We consider the function $L(\epsilon)$, as in [51], for any $u \in W_B^{1,2}$ fixed,

$$L(\epsilon) = L(w_0+\epsilon u) = \log\int_M (w_0+\epsilon u)^2\tau^{-n} + \left(\int_M (w_0+\epsilon u)^2\tau^{-n}\right)^{-1}\mathcal{W}(g, w_0+\epsilon u, \tau).$$

Then it is straightforward to check that $L(\epsilon)$ has a minimum at $\epsilon = 0$. Taking derivative of $L(\epsilon)$ at $\epsilon = 0$, then we get, for any $u \in W_B^{1,2}$,

$$\int_M (w_0\log w_0^2 - R^T w_0 + \mu(g,\tau)w_0)u - 4\langle\nabla w_0, \nabla u\rangle = 0. \tag{9.5}$$

If we assume w_0 is smooth (hence $w_0 \in C_B^\infty$), then we get

$$\int_M (4\triangle w_0 + w_0\log w_0^2 - R^T w_0 + \mu(g,\tau)w_0)u = 0$$

for any $u \in W_B^{1,2}$, in particular, for any $u \in C_B^\infty$. Note that if $v \in C_B^\infty$ such that $\int_M uv = 0$ for any $u \in C_B^\infty$, then $v = 0$. Hence if w_0 is smooth, then it follows that w_0 satisfies (4.30). If w_0 is not smooth, we consider the equation

$$\triangle h = 1/4(w_0\log w_0^2 - R^T w_0 + \mu(g,\tau)w_0) := A_0.$$

It is clear that there exists a unique solution (up to addition of a constant) since we can get, by taking $u = 1$ in (9.5),

$$\int_M w_0\log w_0^2 - R^T w_0 + \mu(g,\tau)w_0 = 0.$$

By an approximation argument, it is clear that $h \in W_B^{1,2}$ since $w_0 \in W_B^{1,2}$ and $R^T \in C_B^\infty$. Actually let $w_k \to w_0$ in $W^{1,2}$ such that $w_k \in C_B^\infty$, then we would get

a sequence of solution of $h_k \in C_B^\infty$ such that

$$\triangle h_k = 1/4(w_k \log w_k^2 - R^T w_k + \mu(g,\tau)w_k) := A_k.$$

By Sobolev inequality, we can assume that $A_k \to A_0$ in L^p, $p = 2(2n+1)/(2n-1)-\epsilon$, for any small $\epsilon > 0$. We can assume, for each k,

$$\int_M h = \int_M h_k = 0.$$

Then apply the standard elliptic regularity theory to $\triangle(h-h_k) = A_0 - A_k$, we know that

$$\|h - h_k\|_{W^{2,p}} \leq C\|A_0 - A_k\|_{L^p}.$$

In particular, $h_k \to h$ in $W^{2,p}$, hence $h \in W_B^{1,2}$. Now using (9.5) again, we can get that for any $u \in C_B^\infty$,

$$\int_M (h - w_0)\triangle u = 0.$$

We claim that $h - w_0$ is a constant. Let $u_k \in C_B^\infty$ be a sequence of smooth function such that $u_k \to h - w_0$ in $W^{1,2}$. Then we get that

$$\int_M (h - w_0)\triangle u_k = 0.$$

It then follows

$$\int_M \langle \nabla(h - w_0), \nabla u_k\rangle = 0.$$

Hence we have, letting $k \to \infty$,

$$\int_M |\nabla(h - w_0)|^2 = 0.$$

Hence $h - w_0$ is a constant and w_0 satisfies

$$4\triangle w_0 = w_0 \log w_0^2 - R^T w_0 + \mu(g,\tau)w_0. \tag{9.6}$$

It then follows the well-know De Giogi-Nash theory that $w_0 \in L^\infty$, and then by L^p theory, we can get the further regularity $w_0 \in W^{2,p}$ for any $p > 1$. Furthermore, once we show that w_0 actually satisfies (9.6) almost everywhere, we can apply Lemma on page 114 ([52]) to get that $w_0 > 0$. Rothaus only stated this result in [52] for minimizers of (4.28) in $W^{1,2}$; but his proof only requires the equation (9.6). Hence we can get $w_0 > 0$, and then it follows that w_0 is smooth. □

Remark 9.8. *In the present paper, we actually do not really need the fact that w_0 is positive and smooth.*

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